

GLSM's for Partial Flag Manifolds

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In this paper we outline some aspects of nonabelian gauged linear sigma models. First, we review how partial flag manifolds (generalizing Grassmannians) are described physically by nonabelian gauged linear sigma models, paying attention to realizations of tangent bundles and other aspects pertinent to $(0,2)$ models. Second, we review constructions of Calabi-Yau complete intersections within such flag manifolds, and properties of the gauged linear sigma models. We discuss a number of examples of nonabelian GLSM's in which the Kähler phases are not birational, and in which at least one phase is realized in some fashion other than as a complete intersection, extending previous work of Hori-Tong. We also review an example of an abelian GLSM exhibiting the same phenomenon. We tentatively identify the mathematical relationship between such non-birational phases, as examples of Kuznetsov's homological projective duality. Finally, we discuss linear sigma model moduli spaces in these gauged linear sigma models. We argue that the moduli spaces being realized physically by these GLSM's are precisely Quot and hyperquot schemes, as one would expect mathematically.

April 2007

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1 Introduction

Two-dimensional gauged linear sigma models [1] have proven to be an extremely powerful tool for describing a wide variety of spaces. Their primary original use was for describing toric varieties and complete intersections therein. More recently [2] they have been used to describe toric stacks and complete intersections therein.

In addition to toric manifolds, they have also been used to describe Grassmannians [1, 3], and although this application was comparatively obscure, recently it has been explored in more depth [4]. In this paper we shall push this direction further by outlining basic aspects of gauged linear sigma models for partial flag¹ manifolds, which generalize Grassmannians.

We begin in section 2 with an overview of how flag manifolds are realized in physics via nonabelian gauged linear sigma models, also describing various dualities that exist among the flag manifolds, Plücker coordinates for flag manifolds, and more complicated manifolds and stacks realizable with gauged linear sigma models. In section 3 we describe how tangent bundles of flag manifolds are realized physically in gauged linear sigma models, a necessary step for considering (0,2) theories.

Next, in section 4 we discuss Calabi-Yau complete intersections in flag manifolds. As in [4], we also examine Kähler phases of some gauged linear sigma models, and describe a number of examples in which the different Kähler phases have geometric interpretations, sometimes realized in novel fashions, but are not related by birational transformations. In the past, it was thought that geometric phases of gauged linear sigma models must be related by birational transformations, but recently in [5][section 12.2] and [4], counterexamples have appeared. We interpret the different phases as being related by Kuznetsov’s homological projective duality [6, 7, 8], an idea that will be explored much further in [9]. We use that proposal that gauged linear sigma models realize Kuznetsov’s duality to make a proposal for the physical interpretation of Calabi-Yau complete intersections in $G(2, N)$ for N even, which has been mysterious previously. We also briefly outline how such non-birational phase phenomena can appear in abelian GLSM’s, reviewing an example that first appeared in [5][section 12.2] and which will be explored in greater detail (together with many other examples) in [9].

Finally, in section 5 we review how quantum cohomology of Grassmannians and flag manifolds is realized in corresponding GLSM’s, and describe linear sigma model moduli spaces in nonabelian GLSM’s. Physically, the computation of linear sigma model moduli spaces for nonabelian GLSM’s has not been clear, unlike the case of abelian GLSM’s. Mathematically, on the other hand, there are known candidates for what such spaces should be – Quot and hyperquot schemes, specifically – that have been used by mathematicians when studying

¹In this paper, we shall use the terms “partial flag manifold” and “flag manifold” interchangeably. In particular, “flag manifold” should not be interpreted to mean only a complete flag manifold.

quantum cohomology rings of flag manifolds. In this paper, we show in examples that LSM moduli spaces realized by nonabelian GLSM's are precisely Quot and hyperquot schemes, as one would have expected naively, instead of, for example, some different spaces birationally equivalent to Quot and hyperquot schemes.

2 Flag manifolds

2.1 Basics

A flag manifold is a manifold describing possible configurations of a series of nested vector spaces. We will use the notation $F(k_1, k_2, \dots, k_n, N)$ to indicate configuration space of all k_1 -dimensional planes sitting inside k_2 -dimensional planes which themselves sit inside k_3 -dimensional planes, and so forth culminating in k_n -dimensional planes sitting inside the vector space \mathbf{C}^N .

Grassmannians are a special case of flag manifolds. The Grassmannian of k -planes inside \mathbf{C}^N , denoted $G(k, N)$, is the same as the flag manifold $F(k, N)$. The space $G(k, N)$ has complex dimension $k(N - k)$, and Euler characteristic

$$\binom{N}{k} = \frac{N!}{k!(N - k)!}$$

As one special case, $G(1, N)$ is the projective space \mathbf{P}^{N-1} , so Grassmannians themselves generalize projective spaces.

Flag manifolds can be described as cosets as follows:

$$F(k_1, k_2, \dots, k_n, N) = \frac{U(N)}{U(k_1) \times U(k_2 - k_1) \times \dots \times U(k_n - k_{n-1}) \times U(N - k_n)} \quad (1)$$

which has complex dimension

$$\sum_{i=1}^n (k_i - k_{i-1})(N - k_i) = \sum_{i=1}^n k_i (k_{i-1} - k_i) + Nk_n$$

in conventions where $k_0 = 0$. In particular, as a special case,

$$G(k, N) = \frac{U(N)}{U(k) \times U(N - k)}$$

which has complex dimension $k(N - k)$.

The Euler characteristic of $F(k_1, \dots, k_n, N)$ is given by the multinomial coefficient

$$\frac{N!}{k_1!(k_2 - k_1)! \cdots (k_n - k_{n-1})!(N - k_n)!}$$

Flag manifolds have only even-dimensional cohomology, with a basis given by Schubert classes, which correspond to elements of W/W_P (in a description of the flag manifold as G/P for P a parabolic subgroup of G). Thus, $\chi(G/P) = |W|/|W_P|$.

There are natural projection maps to flag manifolds with fewer flags,

$$F(k_1, \dots, k_n, N) \longrightarrow F(k_1, \dots, \hat{k}_i, \dots, k_n, N)$$

(where \hat{k}_i denotes an omitted entry) defined by forgetting the intermediate flags. The projection map above has fiber $G(k_i - k_{i-1}, k_{i+1} - k_{i-1})$. For example, there is a map

$$F(1, 2, n) \longrightarrow F(1, n) = G(1, n)$$

with fiber $G(1, n - 1)$.

These manifolds also have alternative descriptions which make it possible to describe them with gauged linear sigma models. The Grassmannian $G(k, N)$ can be described as

$$G(k, N) = \mathbf{C}^{kN} // GL(k)$$

which corresponds physically to a two-dimensional $U(k)$ gauge theory with N chiral fields in the fundamental of $U(k)$. The D-terms in the supersymmetric gauge theory have the form [1]

$$D_j^i = \sum_{s=1}^N \phi^{is} \bar{\phi}_{js} - r \delta_j^i$$

We can interpret the sum above as the dot product of two vectors in \mathbf{C}^N , and so for $r \gg 0$, we see that the vectors must be orthogonal and normalized. In other words, the D-terms force the ϕ fields to describe a k -dimensional subspace of \mathbf{C}^N , exactly right to describe the Grassmannian of k planes in \mathbf{C}^N .

Similarly, flag manifolds can be described as

$$F(k_1, k_2, \dots, k_n, N) = \left(\mathbf{C}^{k_1 k_2} \times \mathbf{C}^{k_2 k_3} \times \mathbf{C}^{k_3 k_4} \times \cdots \times \mathbf{C}^{k_{n-1} k_n} \times \mathbf{C}^{k_n N} \right) // (GL(k_1) \times GL(k_2) \times \cdots \times GL(k_n))$$

which corresponds physically to a two-dimensional

$$U(k_1) \times U(k_2) \times \cdots \times U(k_n)$$

gauge theory with bifundamentals.

The D-term analysis is similar to that for Grassmannians. The D-terms for the first $U(k_1)$ factor are of the form

$$D_j^i = \sum_{s=1}^{k_2} \phi^{is} \bar{\phi}_{js} - r_1 \delta_j^i$$

identical to those for Grassmannians, and for $r_1 \gg 0$ the conclusion is the same: the ϕ 's define k_1 orthogonal, normalized vectors in \mathbf{C}^{k_2} . A slightly more interesting case is a generic $U(k_i)$ factor. Here the D-terms are of the form

$$D_j^i = - \sum_{s=1}^{k_{i-1}} \phi^{is} \bar{\phi}_{js} + \sum_{a=1}^{k_{i+1}} \tilde{\phi}^{ai} \bar{\tilde{\phi}}_{aj} - r_i \delta_j^i$$

where the ϕ 's are $(\mathbf{k}_{i-1}, \overline{\mathbf{k}}_i)$ bifundamentals and the $\tilde{\phi}$'s are $(\mathbf{k}_i, \overline{\mathbf{k}}_{i+1})$ bifundamentals. If we make the inductive assumption that the ϕ 's define k_{i-1} orthogonal, normalized vectors in \mathbf{C}^{k_i} , then the first sum above is proportional to $r_{i-1} \delta_j^i$ for some i, j , and vanishes for others. So long as both $r_{i-1} \gg 0$ and $r_i \gg 0$, the result is the same: the D-terms give a constraint on the $\tilde{\phi}$, which implies that they are orthogonal, normalized vectors in $\mathbf{C}^{k_{i+1}}$. Proceeding inductively we see that so long as all the Fayet-Iliopoulos parameters are positive, we recover the flag manifold.

In this language, we can define the (forgetful) projection map

$$F(k_1, k_2, \dots, k_n, N) \longrightarrow F(k_1, \dots, \hat{k}_i, \dots, k_n, N)$$

(where \hat{k}_i denotes an omitted entry) as follows. Let B_b^a denote the bifundamentals transforming in the $(\mathbf{k}_{i-1}, \overline{\mathbf{k}}_i)$ representation of $U(k_{i-1}) \times U(k_i)$, and C_c^b denote the bifundamentals transforming in the $(\mathbf{k}_i, \overline{\mathbf{k}}_{i+1})$ representation of $U(k_i) \times U(k_{i+1})$, then the projection map is defined by removing the $U(k_i)$ projection factor and replacing B, C with the factor BC where the internal $U(k_i)$ index is summed over.

For more information on flag manifolds and Grassmannians, see for example [10, 11].

2.2 Plücker coordinates and baryons

Grassmannians and flag manifolds can be embedded into projective spaces using “Plücker coordinates,” which can be understood physically as baryons in the gauged linear sigma model. Let us first review how this works for Grassmann manifolds, then describe the construction for flag manifolds.

Describe a Grassmannian $G(k, N)$ as a $U(k)$ gauge theory with N chiral fields A_s^i transforming in the fundamental representation of $U(k)$. The $SU(k)$ invariant field configurations (“baryons,” for mathematical readers) have the form

$$P_{s_1 \dots s_k} = \epsilon_{i_1 \dots i_k} A_{s_1}^{i_1} A_{s_2}^{i_2} \dots A_{s_k}^{i_k}$$

where i is a $U(k)$ index and $s \in \{1, \dots, N\}$. This gives us

$$\binom{N}{k} = \frac{N!}{k!(N-k)!}$$

$SU(k)$ -invariant composite fields. Since the gauge group is $U(k)$ not $SU(k)$, each of these composite fields transforms with the same weight under the overall $U(1)$, and the zero locus is disallowed by D-terms, we interpret each of these baryons as homogeneous coordinates on a projective space of dimension

$$\binom{N}{k} - 1$$

We can follow a similar procedure to construct Plücker coordinates for flag manifolds. Consider the flag manifold constructed as the classical Higgs moduli space of a

$$U(k_1) \times U(k_2) \times \dots \times U(k_n)$$

gauge theory with bifundamentals in the $(\mathbf{k}_1, \overline{\mathbf{k}}_2)$ representation of $U(k_1) \times U(k_2)$, $(\mathbf{k}_2, \overline{\mathbf{k}}_3)$ representation of $U(k_2) \times U(k_3)$, and so forth, concluding with a bifundamental in the $(\mathbf{k}_n, \overline{\mathbf{N}})$ representation of $U(k_n) \times U(N)$.

Let A_b^a denote the bifundamental of $U(k_1) \times U(k_2)$, B_c^b denote the bifundamental of $U(k_2) \times U(k_3)$, and C_d^c denote the bifundamental of $U(k_n) \times U(N)$. If we contract all internal indices, then the product

$$AB \dots C$$

transforms as the $(\mathbf{k}_1, \overline{\mathbf{N}})$ of $U(k_1) \times U(N)$, and by taking determinants of $k_1 \times k_1$ submatrices, just as for the Grassmannian $G(k_1, N)$, we can build one set of baryons, which we interpret as homogeneous coordinates on a projective space of dimension

$$\binom{N}{k_1} - 1.$$

Contracting internal indices, the product

$$B \dots C$$

transforms as the $(\mathbf{k}_2, \overline{\mathbf{N}})$ representation of $U(k_2) \times U(N)$, and by taking determinants of $k_2 \times k_2$ submatrices, just as for the Grassmannian $G(k_2, N)$, we can build another set of baryons, which we interpret as homogeneous coordinates on a projective space of dimension

$$\binom{N}{k_2} - 1$$

We can continue this process, which at the end will construct the last set of baryons as determinants of $k_n \times k_n$ submatrices of C .

More formally, each product of bifundamentals is defining a (forgetful) map

$$F(k_1, k_2, \dots, k_n, N) \longrightarrow G(k_i, N)$$

for some i . Constructing baryons from a product of bifundamentals is building Plücker coordinates on the Grassmannians, and so ultimately mapping the flag manifold into a product of projective spaces:

$$F(k_1, \dots, k_n, N) \longrightarrow \prod_{i=1}^n \mathbf{P}^{N_i-1}$$

where

$$N_i = \binom{N}{k_i}$$

2.3 Presentation dependence

In gauged sigma models, linear or nonlinear, there is often a question of, or assumptions made concerning, presentation-dependence, which is believed to be resolved by renormalization group flow. For example, the \mathbf{P}^n model can be alternately presented as either an ungauged nonlinear sigma model on \mathbf{P}^n , or, a gauged linear sigma model with $n+1$ matter fields all of charge 1. Mathematically, these are describing the same thing, but physically the two theories are different. In this particular case, it has been checked by a variety of means that these two presentations are in the same universality class.

Such presentation-dependence assumptions in principle often appear in gauged linear sigma models: for example, the supersymmetric \mathbf{P}^1 model and a model describing a degree-two hypersurface in \mathbf{P}^2 are describing the same target space geometry, and so had better be in the same universality class, though we are not aware of any work specifically checking this.

In more complicated cases, presentation-dependence is a more important issue. For example, strings on stacks [2, 5, 12, 13, 14] are described concretely by universality classes of gauged nonlinear sigma models, where the gauge group need be neither finite nor effectively-acting. Because several naive consistency checks fail, a great deal of effort was expended in especially [2, 13] to verify that universality classes are independent of presentation. Also, presentation-dependence issues arise in the application of derived categories to physics [14], though there the presentations involve open strings, not gauged sigma models.

Here let us list some interesting examples of flag manifolds which can be presented both with nonabelian gauge theories and, alternately, with abelian gauge theories.

One example, discussed in [4], is the Grassmannian $G(2, 4)$ of 2-planes in \mathbf{C}^4 , which can be understood as a hypersurface in \mathbf{P}^5 , as follows: let $\phi_{ij} = -\phi_{ji}$, $i \in \{1, \dots, 4\}$, denote

homogeneous coordinates on \mathbf{P}^5 , then $G(2, 4)$ is given by the hypersurface

$$\phi_{12}\phi_{34} - \phi_{13}\phi_{24} + \phi_{14}\phi_{23} = 0$$

In fact, this is just the Plücker embedding: each ϕ_{ij} is a baryon on $G(2, 4)$, and it is straightforward to check that the equation above follows from the definition of Plücker coordinates. Furthermore, this result is invariant under duality transformations.

Another example is the flag manifold $F(1, n-1, n)$, which is the degree $(1, 1)$ hypersurface in $\mathbf{P}^{n-1} \times \mathbf{P}^{n-1}$. This also can be understood as a Plücker embedding. Let $A_a^1, B_i^a, a \in \{1, \dots, n-1\}, i \in \{1, \dots, n\}$ denote the bifundamentals of the two-dimensional gauge theory, then define Plücker coordinates

$$\begin{aligned} p(AB)_{i_1} &= A_a^1 B_{i_1}^a \\ p(B)_{i_1 \dots i_{n-1}} &= B_{i_1}^{a_1} \dots B_{i_{n-1}}^{a_{n-1}} \epsilon_{a_1 \dots a_{n-1}} \end{aligned}$$

Define q_i 's to be the dual of the $p(B)$'s:

$$q_i = \frac{1}{(n-1)!} \epsilon_{ii_1 \dots i_{n-1}} p(B)_{i_1 \dots i_{n-1}}$$

then the image of the Plücker embedding in $\mathbf{P}^{n-1} \times \mathbf{P}^{n-1}$ is the same as the degree $(1, 1)$ hypersurface in $\mathbf{P}^{n-1} \times \mathbf{P}^{n-1}$ given by

$$\sum_i p(AB)_i q_i = 0$$

This follows from the identity that

$$\sum_i A_a^1 B_i^a B_{i_1}^{a_1} \dots B_{i_{n-1}}^{a_{n-1}} \epsilon_{ii_1 \dots i_{n-1}} \epsilon_{a_1 \dots a_{n-1}} = 0$$

The reader should be careful to note that these examples are the exception, not the rule: in general, although Plücker coordinates will define an embedding into a product of projective spaces, typically that embedding cannot be described as a complete intersection of hypersurfaces in those projective spaces. For the embedding to be given globally as a complete intersection, as in the examples above, is rare.

2.4 Duality

The Grassmannian of k -planes in \mathbf{C}^N is the same as the Grassmannian of $N - k$ planes in \mathbf{C}^N :

$$G(k, N) \cong G(N - k, N)$$

Physically, this is a duality between a two-dimensional $U(k)$ field theory with N fundamentals and a two-dimensional $U(N - k)$ field theory with N fundamentals. Since both gauge theories describe the same manifold, following the usual procedure for linear sigma models one assumes that the chiral rings of each theory match. This is very reminiscent of Seiberg duality in four dimensions [15], which relates $SU(k)$ gauge theories with N fundamentals and $SU(N - k)$ gauge theories with N fundamentals, for which much of the original justification came from comparing chiral rings.

Let us briefly consider how Plücker coordinates behave under this duality. If we let p_{i_1, \dots, i_k} denote Plücker coordinates on $G(k, N)$ (where p is antisymmetric in its indices and all take values in $\{1, \dots, N\}$), and $q_{i_1, \dots, i_{N-k}}$ denote Plücker coordinates on $G(N - k, N)$, then it can be shown [16][chapter II.VII.3, theorem 1] that they are related by

$$p_{i_1, \dots, i_k} = \frac{1}{(N - k)!} \epsilon_{i_1, \dots, i_N} q_{i_{k+1}, \dots, i_N}$$

This will be useful when relating Calabi-Yau complete intersections and bundles defined over dual Grassmannians.

For flag manifolds, there is an analogous duality: the manifold of k_1 -planes in k_2 -planes in k_3 -planes and so forth is diffeomorphic to the manifold of k_1 -planes in $(k_3 - k_2 + k_1)$ -planes in k_3 planes:

$$F(k_1, k_2, k_3, \dots, N) \cong F(k_1, k_3 - k_2 + k_1, k_3, \dots, N)$$

More generally, any entry k_i can be replaced by $k_{i+1} - k_i + k_{i-1}$, with the exception of k_1 which dualizes to $k_2 - k_1$. It is straightforward to check that the coset representation (1) is symmetric under this duality. However, although this duality relates flag manifolds by diffeomorphisms, the resulting flag manifolds need not be biholomorphic. In general, this duality can be described explicitly using the metric. A flag of dimensions k_1, \dots, k_n is equivalent to a collection of mutually orthogonal subspaces of dimensions $k_1, k_2 - k_1, \dots$. The duality simply amounts to permuting the order of these subspaces and rearranging them into a new flag.

Let us consider a specific example of this duality in action. The flag manifold $F(1, 2, n)$ is the projectivization of the total space of the tangent bundle to \mathbf{P}^{n-1} , and $F(1, n - 1, n)$ is the projectivization of the cotangent bundle to \mathbf{P}^{n-1} . The projection to \mathbf{P}^{n-1} in both cases is given by the forgetful map to $F(1, n) = \mathbf{P}^{n-1}$, and the fibers of that projection are $F(1, n - 1)$ and $F(n - 2, n - 1)$, respectively, which are the projectivizations of the tangent and cotangent spaces. These two spaces are diffeomorphic: one can be written in the form

$$\frac{U(n)}{U(1) \times U(1) \times U(n - 2)}$$

and the other in the form

$$\frac{U(n)}{U(1) \times U(n - 2) \times U(1)}$$

and there is certainly a diffeomorphism exchanging these two quotient spaces. We can perform a consistency check of the existence of that diffeomorphism by calculating cohomology rings. For a vector bundle $E \rightarrow M$ and rank n , the cohomology ring of the total space of the projectivization of E is given by [17][section 20]:

$$H^*(\mathbf{P}E) = H^*(M)[x]/(x^n + c_1(E)x^{n-1} + \cdots + c_n(E))$$

The cohomology of the total spaces of the projectivization of the tangent and cotangent bundles are isomorphic, related in the notation above by sending $x \mapsto -x$. There exists an analogous diffeomorphism relating Hirzebruch surfaces of different degree, though there the diffeomorphism preserves a projectivized complex vector bundle structure, and so the diffeomorphism can be understood by describing each Hirzebruch surface as the projectivization of a vector bundle and arguing that the vector bundles are smoothly isomorphic, modulo tensoring with an overall line bundle. In the present case, by contrast, the diffeomorphism does not preserve the projectivized complex vector bundle structure, and so no such argument can apply. (In fact, such an argument necessarily cannot work. One would need a line bundle L such that $T\mathbf{P}^{n-1} \otimes L \cong T^*\mathbf{P}^{n-1}$ as smooth bundles, but this implies

$$c_1(T\mathbf{P}^{n-1}) + (n-1)c_1(L) = -c_1(T\mathbf{P}^{n-1})$$

or, more simply,

$$(n-1)c_1(L) = -2c_1(T\mathbf{P}^{n-1}) = -2n$$

but no such L can exist, except possibly when $n-1 = 2$ or 1 .)

By repeatedly applying the duality above, one can show that as a special case,

$$F(k_1, \dots, k_n, N) \cong F(N - k_n, N - k_{n-1}, \dots, N - k_2, N - k_1, N)$$

Unlike the dualities above, this special case defines a biholomorphism, not just a diffeomorphism.

It is straightforward to determine how Plücker coordinates behave under the biholomorphic duality

$$F(k_1, \dots, k_n, N) \leftrightarrow F(N - k_n, \dots, N - k_1, N)$$

of flag manifolds. Let

$$\begin{aligned} p(AB \cdots C)_{i_1 \cdots i_{k_1}} \\ p(B \cdots C)_{i_1 \cdots i_{k_2}} \\ p(C)_{i_1 \cdots i_{k_n}} \end{aligned}$$

denote Plücker coordinates formed from the baryons $AB \cdots C$, $B \cdots C$, and C on the flag manifold $F(k_1, \dots, k_n, N)$ above. On the dual flag manifold, let

$$\begin{aligned} q(AB \cdots C)_{i_1 \cdots i_{N-k_n}} \\ q(B \cdots C)_{i_1 \cdots i_{N-k_{n-1}}} \\ q(C)_{i_1 \cdots i_{N-k_1}} \end{aligned}$$

be their analogues. Then, essentially² as a consequence of the relation between Plücker coordinates on dual Grassmannians, the current sets of Plücker coordinates are related in the form

$$p(AB \cdots C)_{i_1 \cdots i_{k_1}} = \frac{1}{(N - k_1)!} \epsilon_{i_1 \cdots i_N} q(C)_{i_{k_1+1} \cdots i_N}$$

In terms of the corresponding two-dimensional gauge theory, the general duality means that a given $U(k_i)$ factor can be replaced by a $U(k_{i+1} - k_i + k_{i-1})$ factor, at the same time that bifundamentals are also replaced, and because the flag manifolds are the same, for consistency of linear sigma model presentations one assumes that the chiral rings of these two two-dimensional gauge theories also match.

This two-dimensional duality is very reminiscent of duality cascades in four-dimensional gauge theories [18, 19]. In particular, in the two-dimensional gauge theory, the $U(k_i)$ factor has $k_{i+1} + k_{i-1}$ fundamentals, and so applying Seiberg duality on that factor would replace the $U(k_i)$ by $U(k_{i+1} - k_i + k_{i-1})$. Here, however, the analogy with Seiberg duality begins to break down, because dualities on each individual i do not typically yield biholomorphisms, merely diffeomorphisms of the target space, and unlike biholomorphisms, diffeomorphisms need not preserve the chiral ring.

2.5 Weighted Grassmannians

There exists a notion of ‘weighted Grassmannians’ [20], which generalizes both ordinary Grassmannians and weighted projective spaces. Let us briefly review their construction and physical realization.

Briefly, the idea is to either modify the group $GL(k)$ ($U(k)$) appearing in the GIT (symplectic) quotient construction of the Grassmannian $G(k, N)$ with another group, and/or modify its action on the N fundamentals.

Before describing the weighted Grassmannian, let us describe the affine Grassmannian, as a simple prototype for these constructions. The affine Grassmannian corresponding to the ordinary Grassmannian $G(k, N)$ is defined by the GIT quotient

$$\mathbf{C}^{kN} // SL(k)$$

– in other words, we replace $GL(k)$ by $SL(k)$. The $SL(k)$ acts by sending a $k \times N$ matrix A to SA , for $S \in SL(k)$. Physically, this is realized by an $SU(k)$ gauge theory with N fundamentals, instead of a $U(k)$ gauge theory with N fundamentals.

²Contractions such as $AB \cdots C$ implicitly define projections to Grassmannians, so the duality described here for Plücker coordinates on flag manifolds is, in fact, an immediate consequence of the relation between Plücker coordinates for dual Grassmannians.

A weighted Grassmannian can be constructed as a \mathbf{C}^\times quotient of the affine Grassmannian. However, that \mathbf{C}^\times action does not lift to an action on \mathbf{C}^{kN} – a weighted Grassmannian is not, $\mathbf{C}^{kN}/(SL(k) \times \mathbf{C}^\times)$. Instead, to lift to an action on \mathbf{C}^{kN} , to describe the weighted Grassmannian as a quotient of \mathbf{C}^{kN} and so as a nonabelian gauge theory, we must work a bit harder. Specifically, we must replace \mathbf{C}^\times by a cover of \mathbf{C}^\times , call it $\widetilde{\mathbf{C}}^\times$, which *will* lift to an action on \mathbf{C}^{kN} , and then we must quotient out a noneffectively-acting part of the $SL(k) \times \widetilde{\mathbf{C}}^\times$ quotient.

The next step is to construct the group by which we will be quotienting \mathbf{C}^{kN} to recover the weighted Grassmannian, not just the ordinary Grassmannian. In fact, we will see there are several closely related groups, all slightly different quotients of $SL(k) \times \widetilde{\mathbf{C}}^\times$.

First, let us define $\widetilde{\mathbf{C}}^\times$. This is an extension of \mathbf{C}^\times by \mathbf{Z}_k :

$$1 \longrightarrow \mathbf{Z}_k \longrightarrow \widetilde{\mathbf{C}}^\times \longrightarrow \mathbf{C}^\times \longrightarrow 1$$

As a group, $\widetilde{\mathbf{C}}^\times \cong \mathbf{C}^\times$, but occasionally we need to distinguish them in order to precisely describe when a \mathbf{C}^\times has a well-defined action and when we have to replace it by a cover. To be precise, and to hopefully help reduce confusion, we will distinguish the two groups.

Next, let us define the gauge group. This will be denoted G_u , and is defined by

$$G_u = \frac{SL(k) \times \widetilde{\mathbf{C}}^\times}{\mathbf{Z}_k}$$

where the quotient is by the subgroup

$$\left\{ (\zeta^u I, \zeta^{-1}) \in SL(k) \times \widetilde{\mathbf{C}}^\times \mid \zeta^k = 1 \right\}$$

for an integer u . For different integers, we get different groups. For example, when $u = 0$, $G_0 = SL(k) \times \mathbf{C}^\times$. For example, when $u = 1$, $G_1 = GL(k)$. These groups all have the same Lie algebra, but differ globally by finite group factors.

Next, we need to describe how the group G_u acts on \mathbf{C}^{kN} . To do so, we will describe how the cover $SL(k) \times \widetilde{\mathbf{C}}^\times$ acts on \mathbf{C}^{kN} . That action will have a \mathbf{Z}_k kernel, and so will descend to an action of G_u . So, let A be a $k \times N$ matrix. Let $S \in SL(k)$, and define a diagonal $N \times N$ matrix D to have entries μ^{u+kw_i} on the diagonal, for $\mu \in \widetilde{\mathbf{C}}^\times$. Then, $SL(k) \times \widetilde{\mathbf{C}}^\times$ acts as follows:

$$A \mapsto SAD$$

This action has a \mathbf{Z}_k center – the action of the center of $SL(k)$ is equivalent to multiplying on the right by a diagonal matrix. In particular, for all u , this action descends to an action of G_u .

The weighted Grassmannian with weights (u, w_1, \dots, w_N) is then the GIT quotient

$$\mathbf{C}^{kN} // G_u$$

with the G_u action above. In the case $u = 1$ and all $w_i = 0$, the weighted Grassmannian reduces to the ordinary Grassmannian $G(k, N)$. In the special case $k = 1$, the weighted Grassmannian is the same as the weighted projective stack $\mathbf{P}_{[u+w_1, \dots, u+w_N]}^{N-1}$.

Physically, this quotient is realized by a \widetilde{G}_u gauge theory, where \widetilde{G}_u is a quotient of $SU(k) \times U(1)$ by a \mathbf{Z}_k center of the same form as for G_u . This gauge theory has N chiral superfields transforming in k -dimensional representations of G_u , defined by the group action defined above on \mathbf{C}^{kN} .

One can define baryons / Plücker coordinates in the same fashion described earlier, as $SL(k)$ -invariant field combinations. Instead of defining an embedding into an ordinary projective space, Plücker coordinates on a weighted Grassmannian now define an embedding into a weighted projective space or stack, whose weights are given by $u + \sum(w_{\sigma(i)})$, summing over all combinations of k elements of the N basis elements of \mathbf{C}^N .

We can also now shed some light on some earlier comments. We claimed previously that a weighted Grassmannian is a \mathbf{C}^\times quotient of an affine Grassmannian. That \mathbf{C}^\times acts in such a way that the Plücker coordinates have weights $u + \sum w_{\sigma(i)}$. To lift that action to an action on \mathbf{C}^{kN} , each of the N sets of chiral superfields transforming as the \mathbf{k} of $SL(k)$ would have to have weight $(u/k) + w_i$ under the \mathbf{C}^\times . Unless $u = 0$ or is a multiple of k , that does not make sense. Instead, we replace \mathbf{C}^\times with a k -fold cover, whose weights are $u + kw_i$, and then one gets sensible results.

One can also define weighted flag manifolds in an analogous fashion, though we shall not do so here.

2.6 Mixed examples

In this note we have studied gauged linear sigma models describing flag manifolds. It is also possible to mix flag manifolds and toric varieties and stacks. For completeness, we shall very briefly outline a few examples here.

2.6.1 \mathbf{P}^1 bundle over flag manifold

Let us describe a \mathbf{P}^1 bundle over a flag manifold

$$F(k_1, \dots, k_n, N)$$

Begin with a gauged linear sigma model for the flag manifold above, *i.e.* a $U(k_1) \times \dots \times U(k_n)$ gauge theory with bifundamental matter, and add two more chiral superfields p_0, p_1 . Let p_1 be neutral under the determinants of each $U(k_i)$, but let p_0 be charged under those same

determinants. Then, if we gauge an additional $U(1)$ that rotates p_0, p_1 by the same phase factors, then we have a \mathbf{P}^1 bundle over the flag manifold, built as a projectivization $\mathbf{P}\mathcal{E}$ of a rank two vector bundle $\mathcal{E} = \mathcal{O} + L$ on the flag manifold.

Similarly, one can fiber other toric varieties and stacks over a flag manifold.

2.6.2 Gerbe on a flag manifold

Let us describe a \mathbf{Z}_k gerbe over the flag manifold $F(k_1, \dots, k_n, N)$ above. Begin as above with a gauged linear sigma model for the flag manifold, then add one extra chiral superfield p with charge q_i under $\det U(k_i)$. Now, gauge an additional $U(1)$ that acts solely on p , with charge m .

The D-term for the additional $U(1)$ has the form

$$k|p|^2 = r$$

and so when $r \gg 0$, we see that $p \neq 0$, and so p is describing a \mathbf{C}^\times bundle over the flag manifold. Gauging $U(1)$ rotations (in the supersymmetric theory) removes all physical degrees of freedom along the fiber directions. Giving p charge m rather than charge 1 means that we are ‘overgauging’, gauging m rotations rather than a single rotation. This distinction is nonperturbatively meaningful, as discussed in [2], and the resulting low-energy theory is a sigma model on a \mathbf{Z}_m gerbe over the flag manifold, with characteristic class

$$(q_1 \bmod m, q_2 \bmod m, \dots, q_n \bmod m)$$

2.6.3 Grassmannian bundle over \mathbf{P}^1

The simplest possible example of a fibered flag manifold is a \mathbf{P}^1 bundle over \mathbf{P}^1 , *i.e.* Hirzebruch surfaces. In terms of gauged linear sigma models, if we let u, v be homogeneous coordinates on the base and s, t be homogeneous coordinates on the fibers, then we can build such Hirzebruch surfaces as $U(1)^2$ gauge theories with charges

u	v	s	t
1	1	n	0
0	0	1	1

Such notions can be easily generalized. For example, we can describe a Grassmannian $G(k, N)$ bundle over \mathbf{P}^1 , as follows. This will be a $U(1) \times U(k)$ gauge theory, with matter fields u, v (forming homogeneous coordinates on the base \mathbf{P}^1) and ϕ^{is} ($i \in \{1, \dots, k\}$, $s \in \{1, \dots, N\}$, forming the Grassmannian fiber). The $U(k)$ only acts on the ϕ^{is} , which

transform in N copies of the fundamental representation. The $U(1)$ acts on u, v with charge 1, and also acts on the ϕ^{is} with weight p_s , where p_s is a sequence of N integers. (The local $U(k)$ gauge symmetry is preserved but the global $U(N)$ symmetry is broken.) More invariantly, we can think of this as a bundle with fibers $G(k, \mathcal{E})$ where

$$\mathcal{E} = \oplus_{s=1}^N \mathcal{O}_{\mathbf{P}^1}(p_s)$$

More general examples are also certainly possible, though this should suffice for illustrative purposes.

3 Bundles

3.1 Tangent bundles and the (2,2) locus

First, let us recall how the tangent bundle is described by left-moving fermions in the linear sigma model for \mathbf{P}^{N-1} . Recall that \mathbf{P}^{N-1} is described by a theory of N chiral fields in which a $U(1)$ action has been gauged. The action for the gauged linear sigma model has interaction terms of the form

$$\bar{\phi}_i \psi_-^i \lambda_+ - \bar{\phi}_i \psi_+^i \lambda_- + c.c.$$

where ϕ are the bosonic parts of the N chiral multiplets, ψ their superpartners, and λ part of the gauge multiplet. When ϕ has a nonzero vev, we see a linear combination of ψ 's and λ 's becomes massive. To be specific, consider the term containing ψ_+ , and consider the subset of ψ_+^i 's defined by $\psi_+^i = \psi \phi_i$ for some Grassmann-valued parameter ψ . Using the D-term condition

$$\sum_i |\phi_i|^2 = r$$

we see that the Yukawa coupling involving this particular ψ_+^i reduces to

$$\bar{\phi}_i \psi_+^i \lambda_- = \psi \lambda_- |\phi_i|^2 = r \psi \lambda_-$$

Thus, it is the $\psi \phi_i$ combination that becomes massive. Put another way, identifying the ψ_+^i 's with local sections of $\mathcal{O}(1)^{\oplus N}$, we see that the remaining massless ψ_+^i 's couple to the cokernel T in the short exact sequence below:

$$0 \longrightarrow \mathcal{O} \xrightarrow{\otimes \phi_i} \mathcal{O}(1)^{\oplus N} \longrightarrow T \longrightarrow 0$$

but that cokernel T is the tangent bundle of \mathbf{P}^{N-1} , so we see that the remaining massless ψ^i 's couple to the tangent bundle.

Mathematically, on \mathbf{P}^{N-1} we have the “universal subbundle” (or “tautological bundle”) S , of rank one and $c_1(S) = -1$, specifically $\mathcal{O}(-1)$, and the “universal quotient bundle” Q

of rank $N - 1$ and $c_1(Q) = +1$. These are related by the short exact sequence

$$0 \longrightarrow S \longrightarrow \mathcal{O}^N \longrightarrow Q \longrightarrow 0$$

The tangent bundle of the projective space is given by $S^\vee \otimes Q$, and so fits into the short exact sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(1)^{\oplus N} \longrightarrow T\mathbf{P}^{N-1} \longrightarrow 0$$

known as the “Euler sequence.”

Projective spaces are special examples of Grassmannians – specifically,

$$\mathbf{P}^{N-1} = G(1, N)$$

in our notation. Thus, it should not be surprising that the analysis above generalizes easily to Grassmannians.

Let us begin our description of Grassmannians by working through the relevant mathematics. On any Grassmannian $G(k, N)$, we have the “universal subbundle” S , of rank k and $c_1(S) = -1$, defined at any point on the Grassmannian to have fiber defined by the k -dimensional subspace of \mathbf{C}^N corresponding to that point. In addition, we have the “universal quotient bundle” Q of rank $N - k$ and $c_1(Q) = +1$. Let V denote the trivial rank N bundle on $G(k, N)$, then these are related by the short exact sequence

$$0 \longrightarrow S \longrightarrow V \longrightarrow Q \longrightarrow 0$$

Under the duality operation $G(k, N) \leftrightarrow G(N - k, N)$, the universal subbundle S is exchanged with Q^\vee , the dual of the universal quotient bundle, and *vice-versa*. The tangent bundle $T = \text{Hom}(S, Q) = S^\vee \otimes Q$, and so can be described as the cokernel

$$0 \longrightarrow S^\vee \otimes S \longrightarrow S^\vee \otimes V \longrightarrow T \longrightarrow 0$$

(For later use, note that using multiplicative properties of Chern characters ($\text{ch}(T) = \text{ch}(S^\vee)\text{ch}(Q)$) one can immediately show $c_1(T) = N$.)

The analysis of the physics for Grassmannians also proceeds much as for projective spaces. The lagrangian of the corresponding nonabelian GLSM contains Yukawa couplings of the form

$$\overline{\phi}_{is} \lambda_{-j}^i \psi_+^{js}$$

(plus complex conjugates and other chiralities) where i is a $U(k)$ index and $s \in \{1, \dots, N\}$. Thus, some combination of λ_- ’s and ψ_+ ’s will become massive. The ψ_+ ’s that become massive can be understood by making the ansatz $\psi_+^{is} = \psi \phi_{is}$, then applying the D-terms we see

$$\overline{\phi}_{is} \lambda_{-j}^i \psi_+^{js} = \overline{\phi}_{is} \lambda_{-j}^i \phi^{js} \psi = r \delta_i^j \lambda_{-j}^i \psi = r \lambda_{-i}^i \psi$$

and so we see that it is the $\psi\phi_{is}$ combination of ψ_+ 's that becomes massive. Put another way, the remaining massless ψ_+ 's couple to the cokernel T of a short exact sequence of the form

$$0 \longrightarrow S^\vee \otimes S \xrightarrow{\otimes \phi_{is}} S^\vee \otimes V \longrightarrow T \longrightarrow 0$$

and so we realize the tangent bundle T of the Grassmannian in physics.

The tangent bundle of a partial flag manifold can be realized similarly. As for Grassmannians, let us first describe the tangent bundle mathematically, then we shall describe how it is realized physically in the corresponding nonabelian GLSM.

Mathematically, the main difference between the tangent bundle of a Grassmannian and that of a more general partial flag manifold is that instead of a single universal subbundle and universal quotient bundle, we now have a flag of both. First, over $F(k_1, \dots, k_n, N)$ there is a universal flag of subbundles

$$S_1 \hookrightarrow S_2 \hookrightarrow \dots \hookrightarrow S_n \hookrightarrow V$$

where the rank of S_i is k_i , and V is the trivial rank N bundle over the flag manifold. (The fibers of these subbundles are defined at any point on the flag manifold to be the flag corresponding to that point.) In addition, there is a dual flag of quotient bundles

$$V \longrightarrow Q_1 \longrightarrow Q_2 \longrightarrow \dots \longrightarrow Q_n$$

where each of the maps above is onto, and the Q_i 's are defined by the short exact sequences

$$0 \longrightarrow S_i \longrightarrow V \longrightarrow Q_i \longrightarrow 0$$

Furthermore, the classical cohomology ring of the flag manifold is generated by the Chern classes of the quotients S_i/S_{i-1} .

The tangent bundle is constructed as a sequence of extensions of bundles T_i defined as the following cokernels:

$$0 \longrightarrow S_i^\vee \otimes S_i \longrightarrow S_i^\vee \otimes S_{i+1} \longrightarrow T_i \longrightarrow 0$$

in conventions where $S_{n+1} = V$. Note that as a special case, when $n = 1$, so that the flag variety reduces to a Grassmannian, the tangent bundle T_1 is the same as that of the Grassmannian. Now, as a smooth bundle, the tangent bundle of the flag manifold is isomorphic to the direct sum over the T_i 's, though the holomorphic structure is a bit more complicated. Holomorphically, this is an extension rather than a direct sum. Consider the partial flag varieties $F_i = F(k_i, \dots, k_n, N)$, and let P_i be the pullback to F of the tangent bundle to F_i .

This gives a series of extensions relating the intermediate bundles P_i :

$$\begin{array}{ccccccc}
& & P_n = T_n & & & & \\
0 & \longrightarrow & T_{n-1} & \longrightarrow & P_{n-1} & \longrightarrow & P_n \longrightarrow 0 \\
0 & \longrightarrow & T_{n-2} & \longrightarrow & P_{n-2} & \longrightarrow & P_{n-1} \longrightarrow 0 \\
& & \vdots & & & & \\
0 & \longrightarrow & T_1 & \longrightarrow & P_1 & \longrightarrow & P_2 \longrightarrow 0 \\
& & P_1 = TF & & & &
\end{array}$$

where TF is the tangent bundle of the flag manifold. More formally, the T_i form the associated graded bundles to the tangent bundle, and the extensions above reconstruct the tangent bundle from its associated graded.

The tangent bundle TF of the flag manifold can also be described more compactly as the cokernel of the short exact sequence

$$0 \longrightarrow \bigoplus_{i=1}^n S_i^\vee \otimes S_i \xrightarrow{*} \bigoplus_{i=1}^n S_i^\vee \otimes S_{i+1} \longrightarrow TF \longrightarrow 0$$

where the map $*$ consists of the bifundamentals $\phi : S_i \rightarrow S_{i+1}$ on the diagonal and $\phi^\vee : S_{i+1}^\vee \rightarrow S_i^\vee$ just above the diagonal. For example, for the flag manifold $F(k_1, k_2, N)$, the map $*$ is given by

$$\begin{bmatrix} \phi_{12} & \phi_{12}^\vee \\ 0 & \phi_{23} \end{bmatrix}$$

where $\phi_{ij} : S_i^\vee \otimes S_i \rightarrow S_i^\vee \otimes S_j$ is the map induced by the bifundamental mapping $S_i \rightarrow S_j$, and $\phi_{ij}^\vee : S_j^\vee \otimes S_j \rightarrow S_i^\vee \otimes S_j$ is the map induced by the dual bifundamental $S_j^\vee \rightarrow S_i^\vee$. In other words,

$$\begin{aligned}
(\phi_{12}) : S_1^\vee \otimes S_1 &\longrightarrow S_1^\vee \otimes S_2 \\
(\phi_{12}^\vee + \phi_{23}) : S_2^\vee \otimes S_2 &\longrightarrow (S_1^\vee \otimes S_2) \oplus (S_2^\vee \otimes S_3)
\end{aligned}$$

Similarly, for the flag manifold $F(k_1, k_2, k_3, N)$, the map $*$ is given by

$$\begin{bmatrix} \phi_{12} & \phi_{12}^\vee & 0 \\ 0 & \phi_{23} & \phi_{23}^\vee \\ 0 & 0 & \phi_{34} \end{bmatrix}$$

We see that the quotient TF is filtered but, since the maps are not block diagonal, it does not decompose as a direct sum. It can be deformed to the direct sum of the T_i by setting the off-diagonal terms to 0.

Next, we shall describe how this structure is realized physically in a nonabelian GLSM describing the partial flag manifold. Consider the flag manifold $F(k_1, \dots, k_n, N)$, realized as

a two-dimensional $U(k_1) \times \cdots \times U(k_n)$ gauge theory with bifundamentals. First consider the $U(k_1)$ factor. There are Yukawa couplings of the form

$$\bar{\phi}_{is} \lambda_{-j}^i \psi_+^{js}$$

(where s is a $U(k_2)$ index), which can be analyzed in exactly the same way as those for a Grassmannian. The combination $\psi_+^{is} = \psi \phi^{is}$ becomes massive. Next, suppose ψ_+^{is} are the fermionic components of a chiral superfield in the $(\mathbf{k}_{i-1}, \bar{\mathbf{k}}_i)$ representation of $U(k_{i-1}) \times U(k_i)$, and $\tilde{\psi}_+^{ai}$ are the corresponding components of a chiral superfield in the $(\mathbf{k}_i, \bar{\mathbf{k}}_{i+1})$ representation of $U(k_i) \times U(k_{i+1})$. Here, the Yukawa couplings involving the $U(k_i)$ λ_- 's are of the form

$$-\bar{\phi}_{is} \lambda_{-j}^i \psi_+^{js} + \bar{\phi}_{ai} \lambda_{-j}^i \tilde{\psi}_+^{aj}$$

If we make the ansatz $\psi_+^{is} = \psi \phi^{is}$ and $\tilde{\psi}_+^{ai} = \psi \tilde{\phi}^{ai}$, for the same Grassmann parameter ψ in both cases, then using the D-terms the Yukawa couplings above reduce to

$$\left(-\bar{\phi}_{is} \phi^{js} + \bar{\phi}_{ai} \tilde{\phi}^{aj} \right) \lambda_{-j}^i \psi = r_i \text{Tr } \lambda_- \psi$$

Putting this together, we see that the remaining massless $\tilde{\psi}_+$'s are described by the cokernel of the short exact sequence

$$0 \longrightarrow \bigoplus_{i=1}^n S_i^\vee \otimes S_i \xrightarrow{*} \bigoplus_{i=1}^n S_i^\vee \otimes S_{i+1} \longrightarrow TF \longrightarrow 0$$

where the $\bigoplus S_i^\vee \otimes S_i$ factor is realized by the gauginos λ , the $\bigoplus S_i^\vee \otimes S_{i+1}$ factor is realized by the fermionic parts ψ of the bifundamentals, and $*$ is the matrix with bifundamentals on and just above the diagonal described earlier. Thus, we see that the physical analysis of the fermions matches the mathematical picture of TF described earlier.

3.2 Aside: homogeneous bundles

For any parabolic subgroup P of a reductive algebraic group G , we can construct a large number of bundles on the coset space G/P by using a representation ρ of P . The total space of the bundle is $G \times_\rho V$, which has a natural projection to G/P , where V is the vector space on which the representation ρ acts. Such bundles are known as “homogeneous bundles.”

Grassmannians and flag manifolds can be described as coset spaces G/P , where in each case $G = GL(N)$, and their tangent bundles are examples of homogeneous bundles. The representations of the relevant parabolic P for a Grassmannian $G(k, N)$ are the same as those of $U(k) \times U(N-k)$, so we merely need to find the relevant representation of $U(k) \times U(N-k)$. The universal subbundle S corresponds to the fundamental representation \mathbf{k} of $U(k)$, and the universal quotient bundle Q corresponds to the dual $\overline{\mathbf{N}-\mathbf{k}}$ of the fundamental representation

of $U(N - k)$. Since the tangent bundle of $G(k, N)$ is given by $\text{Hom}(S, Q)$, we see that the tangent bundle is defined in this fashion by the representation $(\overline{\mathbf{k}}, \overline{\mathbf{N}} - \overline{\mathbf{k}})$ of $U(k) \times U(N - k)$.

The tangent bundle of the flag manifold can be described in an analogous fashion. Each of the bundles S_i can be described by the representation

$$(\mathbf{k}_1, 0, \dots, 0) \oplus (0, \mathbf{k}_2 - \mathbf{k}_1, 0, \dots, 0) \oplus (0, 0, \mathbf{k}_3 - \mathbf{k}_2, 0, \dots, 0) \oplus (0, \dots, 0, \mathbf{k}_i - \mathbf{k}_{i-1}, 0, \dots, 0)$$

of $U(k_1) \times U(k_2 - k_1) \times \dots \times U(N - k_n)$. Call this representation ρ_{S_i} . Similarly, each of the bundles T_i that appeared in the construction of the tangent bundle is homogeneous, and from their definition it should be clear that they are defined by the representation

$$\begin{aligned} \rho_{S_i}^* \otimes (0, \dots, 0, \overline{\mathbf{k}_{i+1} - \mathbf{k}_i}, 0, \dots, 0) \\ = (\overline{\mathbf{k}_1}, 0, \dots, 0, \overline{\mathbf{k}_{i+1} - \mathbf{k}_i}, 0, \dots, 0) \oplus \dots \oplus (0, \dots, 0, \overline{\mathbf{k}_i - \mathbf{k}_{i-1}}, \overline{\mathbf{k}_{i+1} - \mathbf{k}_i}, 0, \dots, 0) \end{aligned}$$

However, typical bundles one works with in a (0,2) model will not be homogeneous, as we shall see in the next section.

3.3 More general bundles

It is straightforward to describe other bundles in (0,2) GLSM's over flag manifolds. For example, one common type of bundle described with (0,2) GLSM's is given as the kernel \mathcal{E} of a short exact sequence:

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{V}_1 \xrightarrow{F} \mathcal{V}_2 \longrightarrow 0$$

The bundles \mathcal{V}_1 and \mathcal{V}_2 are determined by representations ρ_1, ρ_2 of the gauge group. Physically, to describe \mathcal{V}_1 we introduce left-moving fermi multiplets Λ in the ρ_1 representation of the gauge group, and right-moving chiral superfields p in the ρ_2^* representation of the gauge group. The map $\mathcal{V}_1 \rightarrow \mathcal{V}_2$ is realized by a (0,2) superpotential

$$W = \int d\theta p F \Lambda$$

For example, one of the terms descending from that superpotential is

$$\psi_p^s \lambda_-^i F^{is}(\phi)$$

So long as the ϕ 's have nonzero vevs, a linear combination of ψ_p and λ_- will become massive. The subset of λ_- that remain massless are given by the kernel of the map defined by F , hence this superpotential describes a kernel.

For example, consider a bundle on $G(k, N)$ built as a kernel. Let ρ_1 be \mathbf{k} , and ρ_2 be invariant under $SU(k)$, Take

$$F^i = \sum_{s=1}^N \alpha_s \phi^{is}$$

for some arbitrary constants α_i . With the superpotential $\int d\theta p \Lambda F$, we have a (0,2) GLSM on $G(k, N)$ with bundle described by the kernel of F above. Note that this bundle is not homogeneous, and cannot be described in terms of representation theory.

For another example, let us build a bundle on $G(2, N)$ as a kernel. Take ρ_1 to be the $\mathbf{2} \otimes \mathbf{2}$ representation of $U(2)$, and take ρ_2 to be invariant under $SU(2)$. Take the map $F^{ij} = \epsilon^{ij}$. Then our bundle over $G(2, N)$ is defined by the symmetric $(\mathbf{3})$ representation of $U(2)$.

Other standard (0,2) bundle constructions [21] are also possible, but we shall not discuss them further here.

4 Complete intersections in flag manifolds

4.1 Calabi-Yau's in flag manifolds

We can add matter fields p_α to describe hypersurfaces with gauge-invariant superpotentials, following [1, 4]. The condition for the intersection of those hypersurfaces to be Calabi-Yau can be described as the condition for the axial anomaly in the two-dimensional (2, 2) gauged linear sigma model to vanish. In effect, there will be n $U(1)$ factors, corresponding to the determinants of the $U(k_i)$'s, and so n conditions for the intersection to be Calabi-Yau. Following the analysis of [4] in the Grassmannian case, and given that our gauged linear sigma models for flag manifolds have bifundamental matter, it is straightforward to see that the Calabi-Yau condition can be stated as follows:

i	Minus the sum of p_α charges in $\det U(k_i)$ ($= -K_F$)
1	k_2
2	$k_3 - k_1$
3	$k_4 - k_2$
\dots	\dots
n	$N - k_{n-1}$

The right column is determined as the difference between the number of fundamentals and antifundamentals charged under the corresponding $U(k_i)$. The right column can also be understood mathematically as minus the part of the canonical divisor corresponding to that $U(k_i)$ factor.

A list of Calabi-Yau three-folds obtained as complete intersections in flag manifolds is provided at the end of [22]. For completeness, we repeat that list here:

n	F	$\dim(F)$	$-K_F$
7	$F(2, 7)$	10	7
7	$F(1, 2, 7)$	11	(2, 6)
7	$F(1, 5, 7)$	14	(5, 6)
7	$F(1, 2, 6, 7)$	15	(2, 5, 5)*
6	$F(2, 6)$	8	6
6	$F(3, 6)$	9	6
6	$F(1, 2, 6)$	9	(2, 5)
6	$F(1, 3, 6)$	11	(3, 5)
6	$F(1, 4, 6)$	11	(4, 5)
6	$F(1, 2, 5, 6)$	12	(2, 4, 4)
6	$F(1, 3, 5, 6)$	13	(3, 4, 3)
5	$F(2, 5)$	6	5
5	$F(1, 2, 5)$	7	(2, 4)
5	$F(2, 3, 5)$	8	(3, 3)
5	$F(1, 3, 5)$	8	(3, 4)
5	$F(1, 2, 4, 5)$	9	(2, 3, 3)
5	$F(1, 2, 3, 5)$	9	(2, 2, 3)
5	$F(1, 2, 3, 4, 5)$	10	(2, 2, 2, 2)
4	$F(2, 4)$	4	4
4	$F(1, 2, 4)$	5	(2, 3)
4	$F(1, 2, 3, 4)$	6	(2, 2, 2)

(*: This corrects a trivial typo in [22].)

Note that the Calabi-Yau's built in Grassmannians listed in [4][table 2] and [23][section 6.2] form a subset of the list above.

The observant reader will note that, according to our earlier description of dualities,

$$\begin{aligned}
F(1, 3, 6) &\cong F(1, 4, 6) \\
F(1, 3, 5) &\cong F(2, 3, 5) \\
F(1, 2, 4, 5) &\cong F(1, 2, 3, 5)
\end{aligned}$$

should be related by diffeomorphisms. However, although the manifolds are diffeomorphic, note that $-K_F$ differs. This is consistent because $-K_F$ is determined in part by the complex structure, and the diffeomorphisms relating the special cases above do not preserve the complex structure. Put another way, $c_1(K)$ matches c_1 of the holomorphic part of the complexified tangent bundle, but as Chern classes can only be defined for complexified tangent bundles, there is no reason why the c_1 's need be invariant under non-holomorphic

diffeomorphisms. For a simpler example of this principle, the map $z \mapsto \bar{z}$ defines a non-holomorphic diffeomorphism $\mathbf{P}^1 \rightarrow \mathbf{P}^1$, which sends $c_1(K) \mapsto -c_1(K)$.

The analysis also easily extends to the weighted Grassmannians and flag manifolds described in section 2.5. For example, for a weighted Grassmannian modelled on $G(k, N)$ with weights (u, w_1, \dots, w_N) , the Calabi-Yau condition for a complete intersection is that the sum of the degrees of the hypersurfaces must equal

$$Nu + k \sum_i w_i$$

For an ordinary Grassmannian $G(k, N)$, $u = 1$ and all the $w_i = 0$, and so the condition reduces to the constraint that the sum of the degrees equal N .

4.2 Non-birational derived equivalences in nonabelian GLSMs

In [5][section 12.2] and [4], examples were given of GLSM's in which

- one of the Kähler phases had a geometric interpretation, realized in a novel fashion, and
- geometric Kähler phases were not birational

in abelian and nonabelian GLSM's, respectively. It has long been assumed that the geometric phases of GLSM's were related by birational transformations, so examples contradicting that belief are of interest.

In this section we shall elaborate on this matter, by studying further examples of this phenomenon in nonabelian GLSM's, and reviewing an example of the phenomenon in abelian GLSM's. We will also describe a tentative proposal for understanding the mathematical relationship between non-birational phases: we propose that they should be understood in terms of Kuznetsov's homological projective duality [6, 7, 8]. In this paper we will only begin to outline the relevance of Kuznetsov's work – a much more thorough description, and further application to abelian GLSM's, will appear in [9].

We shall begin by reviewing the example in [4], then discuss other examples in non-abelian GLSM's before interpreting the results in terms of Kuznetsov's work and outlining an analogous example in abelian GLSM's.

4.2.1 Hori-Tong-Rodland example

In [4][section 5], an example of a gauged linear sigma model was analyzed which described, for $r \gg 0$, a complete intersection of seven degree one hypersurfaces in the Grassmannian

$G(2, 7)$, and for $r \ll 0$, the “Pfaffian Calabi-Yau,” which is not a complete intersection. (See [24] for a corresponding mathematical discussion.) The superpotential terms each involve two of the chiral fields Φ_i^a defining the Grassmannian and a single auxiliary field p^i , introduced to describe members of the complete intersection, and so has the form

$$W \propto A_i^{jk} p^i \Phi_j^a \Phi_k^b \epsilon_{ab}$$

for constants A_i^{jk} . In terms of the $U(1) \subset U(2)$ given by matrices proportional to the identity, the p^i have charge 2 and the Φ_i^a have charge 1. For $r \gg 0$, it is straightforward to check that the gauged linear sigma model describes $G(2, 7)$ [17]. For $r \ll 0$, the D-terms forbid the p^i from all vanishing, and so they form homogeneous coordinates on a \mathbf{P}^6 . The $U(2)$ gauge symmetry is broken to $SU(2) \times \mathbf{Z}_2$, where the \mathbf{Z}_2 subgroup of $U(2)$ is given by matrices of the form $\text{diag}(1, \zeta)$, for $\zeta = \pm 1$. That \mathbf{Z}_2 acts trivially on the p^i but acts effectively on the Φ_i^a . The matrix $A_i^{jk} p^i \equiv A(p)^{jk}$ is a skew-symmetric 7×7 matrix with entries linear in the p^i . Nondegeneracy of the original complete intersection forces $A(p)$ to have rank either 4 or 6 for all p . At p for which $A(p)$ has rank 4, there are $7 - 4 = 3$ massless Φ doublets; the rest are massive with masses proportional to $|r|$. At p for which $A(p)$ has rank 6, there is $7 - 6 = 1$ massless Φ doublet. As discussed in [4], in the infrared limit all vacua in the nonabelian gauge theory on the rank 6 locus run to infinity, whereas on the rank 4 locus a single vacuum remains. Thus, [4] identify the $r \ll 0$ phase with a nonlinear sigma model on the vanishing locus of the 6×6 Pfaffians of the matrix $A(p)$ defined over \mathbf{P}^6 .

4.2.2 Aside: Pfaffian varieties

As Pfaffian varieties are not commonly used in the physics community, let us take a moment to check certain basic properties of the previous example. (For a more complete description, see for example [11][section 3.5]. For other related information, see for example [25].)

First, let us establish some notation that will sometimes be used elsewhere in this paper. The Pfaffian variety $Pf(N)$ for N odd is a space defined as follows. Let V be an N -dimensional vector space, and consider the locus in the space of two-forms in $\Lambda^2 V^*$ whose rank is not maximal. The maximal possible rank is $N - 1$, and the next possible rank is $N - 3$. So, for N odd, $Pf(N)$ can be defined as the space of skew-symmetric $N \times N$ matrices of rank $N - 3$, *i.e.* whose $(N - 1) \times (N - 1)$ Pfaffians all vanish.

In particular, $Pf(5)$ and $G(2, 5)$ are the same space, but on dual vector spaces. In terms of Plücker coordinates, $G(2, 5)$ describes the space of elements of $\Lambda^2 V$ which have the smallest possible nonzero rank, whereas $Pf(5)$ describes the locus of elements of $\Lambda^2 V^*$ of submaximal rank, which is also 2 in this case. Alternatively, by identifying matrix entries with coordinates on the space of all skew-symmetric matrices and performing a linear change of coordinates, we can describe $Pf(5)$ (and hence $G(2, 5)$) as the vanishing locus in \mathbf{P}^9 (the projectivized space of all skew-symmetric 5×5 matrices, using the fact that such matrices

have ten different entries) of the 4×4 Pfaffians of a generic 5×5 skew-symmetric matrix whose entries are linear in the homogeneous coordinates on \mathbf{P}^9 .

For $N > 5$, the picture is slightly richer. The group $GL(V) = GL(N)$ acts on $\mathbf{P}(\Lambda^2 V^\vee)$ with a finite number of orbits O_{2k} , classified by an even integer $2k \leq N$. Orbit O_{2k} consists of (the projectivizations of) all skew-symmetric matrices A of rank equal to $2k$. (Clearly, any such A can be taken to any other by some element of $GL(V)$.) In general, the Grassmannian is $G(2, V) = O_2$, while the Pfaffian is $Pf = O_{N-3}$ for N odd, and O_{N-1} is the dense open subset of A 's of maximal rank. The orbits are nested: the boundary $\overline{O_{2k}} - O_{2k}$ of the orbit O_{2k} consists precisely of the union of the O_{2i} for $i < k$. For $k < N - 1$, the orbit O_{2k} is singular along its boundary. An easy computation similar to what will be done momentarily for Pf shows that the codimension in $\mathbf{P}(\Lambda^2 V^\vee)$ of O_{N-j} , for even $N - j$, is $j(j - 1)/2$.

Next, let us compute the dimension of the vanishing locus. This is not a global complete intersection, so we cannot get the dimension by subtracting the number of equations. An alternative is to work as follows. At a generic point on \mathbf{P}^6 , the matrix A is rank 6, and so can be block-diagonalized to the form

$$\left[\begin{array}{c|c|c} 0_3 & B_3 & 0 \\ \hline -B_3 & 0_3 & 0 \\ \hline 0 & 0 & 0 \end{array} \right]$$

where B_3 denotes a 3×3 matrix and 0_3 denotes the identically-zero 3×3 matrix. Near a subgeneric point, where all the 6×6 Pfaffians vanish, so that the matrix has rank 4 at the subgeneric point, the matrix A can be block-diagonalized to the form

$$\left[\begin{array}{c|c|ccc} 0_2 & B_2 & 0 & 0 & 0 \\ \hline -B_2 & 0_2 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \alpha_1 & \alpha_2 \\ 0 & 0 & -\alpha_1 & 0 & \alpha_3 \\ 0 & 0 & -\alpha_2 & -\alpha_3 & 0 \end{array} \right]$$

where the α_i are three functions that become zero on the rank 4 locus. Since to get to the locus we must set three functions to zero, we see that our non-global-complete-intersection must have codimension three in \mathbf{P}^6 , and so must have dimension $6 - 3 = 3$. Similarly, the vanishing locus of both the 6×6 Pfaffians as well as the 4×4 Pfaffians is the locus of matrices of rank 2, and working locally as above we find that to get to that locus we would have to locally set $4 + 3 + 2 + 1 = 10$ functions to zero, and so that locus would be codimension 10.

Let us now compute $c_1(T)$ for the vanishing locus in \mathbf{P}^{m-1} of the $(N - 1) \times (N - 1)$ Pfaffians of a skew-symmetric $N \times N$ matrix (N odd) linear in the homogeneous coordinates on \mathbf{P}^{m-1} . (For a more intuitive and less precise way to understand if a given vanishing locus is Calabi-Yau, see for example [11][section 3.5.2].)

Let V be a vector space of odd dimension N . Define $P = \mathbf{P}(\Lambda^2 V^\vee)$, *i.e.*, the space of alternating two-forms on V , modulo scalars. For a generic $A \in P$, the rank of A is $N - 1$.

Now, define

$$Pf \equiv \{A \in P \mid \text{rk } A \leq N - 3\}$$

For $A \in Pf$ the kernel is three dimensional.

Now, define

$$\widetilde{Pf} \equiv \{(A, s) \mid A \in P, As = 0\}$$

This is a partial resolution of Pf . Next, define

$$D \equiv \{A \in P \mid \text{rk } A \leq N - 5\} = \text{singular locus of } Pf$$

and

$$\tilde{D} \equiv \{(A, s) \in \widetilde{Pf} \mid A \in D\}$$

The space \tilde{D} is the exceptional divisor in \widetilde{Pf} , which is blown down to the singular locus D in Pf .

Pulling back the hyperplane bundles via the two composed maps

$$\begin{aligned} \pi : \widetilde{Pf} &\longrightarrow Pf \longrightarrow P = \mathbf{P}(\Lambda^2 V^\vee) \\ \pi' : \widetilde{Pf} &\longrightarrow G(3, V) \longrightarrow \mathbf{P}(\Lambda^3 V) \end{aligned}$$

defines line bundles H, H' , respectively, on \widetilde{Pf} . It can be shown (see below) that the divisor \tilde{D} can be expressed as a linear combination of H, H' , with the result

$$\tilde{D} = ((N - 3)/2)H - H' \tag{2}$$

on \widetilde{Pf} . Since \tilde{D} is blown down in Pf , this linear combination will give a relation between the corresponding divisor classes on Pf :

$$H' = ((N - 3)/2)H \tag{3}$$

on Pf . One can then calculate $c_1(Pf)$ from its embedding into P :

$$c_1(Pf) = c_1(P) - c_1(\mathcal{N}) \tag{4}$$

where \mathcal{N} is the normal bundle to Pf in P . (This is a rank 3 vector bundle away from the singular locus D ; since D has high codimension, one works away from D to get the correct answer for c_1 .) This normal bundle can be identified as

$$\mathcal{N} = \Lambda^2 S^\vee \otimes H \tag{5}$$

where S is (the pullback to $Pf - D = \widetilde{Pf} - \tilde{D}$ of) the universal rank 3 subbundle on $G(3, V)$, fitting into the exact sequence

$$0 \longrightarrow S \longrightarrow V \otimes \mathcal{O}_{G(3, V)} \longrightarrow Q \longrightarrow 0$$

and as above, H is the pullback of $\mathcal{O}_P(1)$. From equation (5) we get that

$$c_1(\mathcal{N}) = 2H' + 3H$$

so from equations (4) and (3) we have

$$\begin{aligned} c_1(Pf) &= (\dim(P) + 1)H - 2H' - 3H \\ &= (\dim(P) - 2)H - 2H' \\ &= (\dim(P) - (N - 1))H \end{aligned}$$

Finally for

$$X \equiv Pf \text{ intersected with a generic linear subspace } \mathbf{P}^{m-1}$$

we get by adjunction

$$\begin{aligned} c_1(TX) &= (\dim(X) - (N - 4))H \\ &= (m - N)H \end{aligned}$$

We still need to justify equation (2) in the cohomology of \widetilde{Pf} . Note that \widetilde{Pf} is a projective bundle over $G(3, V)$, in fact it is the projectivization of $\Lambda^2 Q^\vee$, where as above, Q is the universal quotient bundle over $G(3, V)$. In particular, it follows that $H^2(\widetilde{Pf})$ is two-dimensional, so a relation among H , H' , and \tilde{D} must exist. We can find the actual relation by intersecting with two independent curves a , b in \widetilde{Pf} .

Define a as follows. Fix a 3-space $s \in G(3, V)$, and take a generic pencil of A 's in $\Lambda^2((V/s)^\vee)$, *i.e.* which vanish on this fixed s .

Define b as follows. Take a pencil of A 's which does not intersect D .

In coordinates x_1, \dots, x_N on V , one can take a to be the line in P through the points p , q , and b the line through p , r , where:

$$\begin{aligned} p : & x_{12} + x_{34} + \dots + x_{N-4, N-3} \\ q : & q_{12}x_{12} + \dots + q_{N-4, N-3}x_{N-4, N-3} \\ r : & x_{23} + x_{45} + \dots + x_{N-3, N-2} \end{aligned}$$

with numerical coefficients $q_{12}, \dots, q_{N-4, N-3}$ which are pairwise distinct. The A 's which are linear combinations of p , q vanish on the 3-space s which corresponds to the last three coordinates (*i.e.* is given by the vanishing of x_1, \dots, x_{N-3}), and generically only there, but for the $(N - 3)/2$ parameter values given by the $q_{i, i+1}$ the rank drops to $N = 5$ and the null space contains coordinates $i, i + 1$. The A 's which are linear combinations of p , r have constant rank $N - 3$. The three-dimensional kernel traces a rational normal curve in Plücker space, of degree $(N - 3)/2$. It is now straightforward to count intersection points:

	a	b
H	1	1
H'	0	$(N-3)/2$
\tilde{D}	$(N-3)/2$	0

This establishes our claim that $\tilde{D} = ((N-3)/2)H - H'$, and completes the calculation of $c_1(TX) = (m-N)H$.

4.2.3 A complete intersection in a $G(2, 5)$ bundle

Another analogous example can be built as follows. Let us consider a Calabi-Yau built as a complete intersection in the total space of a $G(2, 5)$ bundle over \mathbf{P}^2 . The $G(2, 5)$ bundle is built by fibering the \mathbf{C}^5 as the vector bundle $\mathcal{O}(-1)^{\oplus 4} \oplus \mathcal{O}$ over \mathbf{P}^2 . The gauged linear sigma model can be built from the fields $x_0, x_1, x_2, \Phi_i^\mu$ ($\mu \in \{1, 2\}, i \in \{1, \dots, 5\}$) in which a $U(2) \times U(1)$ action has been gauged. The $U(2)$ acts on the Φ_i^μ as a set of five doublets, and leaves the x_a invariant. The $U(1)$ charges are as follows:

$$\begin{array}{ccccc} x_0 & x_1 & x_2 & \Phi_{1\dots 4}^\mu & \Phi_5^\mu \\ \hline 1 & 1 & 1 & -1 & 0 \end{array}$$

Baryons (Plücker coordinates) naturally arrange themselves as

$$\begin{array}{ll} \epsilon_{\mu\nu} \Phi_i^\mu \Phi_j^\nu & i, j \in \{1, \dots, 4\} \\ \epsilon_{\mu\nu} \Phi_i^\mu \Phi_5^\nu & i \in \{1, \dots, 4\} \end{array}$$

These baryons define an embedding of the total space of this Grassmannian bundle into the total space of the projective bundle

$$\mathbf{P} \left(\mathcal{O}(-2)^{\oplus 6} \oplus \mathcal{O}(-1)^{\oplus 4} \right) \longrightarrow \mathbf{P}^2$$

We can build a Calabi-Yau by taking the complete intersection of five hypersurfaces, so we add to the gauged linear sigma model five fields p_i of charge -2 under $\det U(2)$ and charge $+1$ under the $U(1)$. The superpotential then has the form

$$W = p_i B^{ijk}(x) \epsilon_{\mu\nu} \Phi_j^\mu \Phi_k^\nu = A^{ij}(x, p) \epsilon_{\mu\nu} \Phi_i^\mu \Phi_j^\nu$$

The 5×5 matrix A^{ij} is skew-symmetric. For $i, j \leq 4$ it is degree $(1, 1)$ in (x, p) , and for $i = 5, j \leq 4$ it is degree $(0, 1)$ in (x, p) .

The analysis of the Landau-Ginzburg point of this model is very similar to that in the previous section. It is given by the vanishing locus of 4×4 Pfaffians of A^{ij} on $\mathbf{P}^2 \times \mathbf{P}^4$, where the \mathbf{P}^4 arises from the p_i fields. Briefly, this is because, as in [4], on the rank 4 locus,

there is a single massless Φ doublet, and the vacua run to infinity, whereas on the rank 2 locus (where the 4×4 Pfaffians vanish), there are three massless Φ doublets, and in the IR a single vacuum remains.

The geometry we have found at the Landau-Ginzburg point was discussed mathematically in [26][section 3.2], as an example of a genus-one-fibered Calabi-Yau threefold, fibered over \mathbf{P}^2 , with an embedding into $\mathbf{P}^2 \times \mathbf{P}^4$ of the form described above. It is not an elliptic fibration, as it does not have an ordinary section, but rather merely a 5-section.

The Landau-Ginzburg geometry above, is believed to have the same derived category of coherent sheaves [26] as the complete intersection in the $G(2, 5)$ bundle³ on \mathbf{P}^2 that sits at the large-radius point in the gauged linear sigma model above. Fiberwise [27], if $G(2, 5)$ is cut by a codimension 5 plane, the result is an elliptic curve, and similarly if the Pfaffian manifold $Pf(5)$ (which can be understood as $G(2, 5)$ again, but describing 2-planes in the dual 5-dimensional vector space) is cut by the dual linear space, the result is the same elliptic curve. The two curves can be identified if one picks a point on each elliptic curve, or alternatively, the second elliptic curve can be identified naturally with Pic^2 of the first curve. When this is done in families, the Grassmannian fibration can be understood as the relative Pic^2 of the Pfaffian fibration, and it is well-known that such pairs are derived equivalent.

Whether or not these two spaces are birational to one another, is not known at present.

4.2.4 A Fano example

In addition to Calabi-Yau examples, we can also consider non-Calabi-Yau examples. In this and subsequent sections, we shall study several examples of nonabelian GLSM's for Fano manifolds. These GLSM's will also possess nontrivial Landau-Ginzburg points. We should note at the beginning, however, that there are some structural differences between GLSM Kähler moduli spaces in Calabi-Yau and non-Calabi-Yau cases. For example, in Calabi-Yau cases the Kähler moduli space is complexified by the theta angle, but in non-Calabi-Yau cases an axial anomaly prevents that theta angle from being meaningful, so the Kähler moduli spaces are real, not complex. In addition, the Kähler parameters will not be renormalization-group invariants in non-Calabi-Yau cases. In the examples below, we will consider some simple nonabelian GLSM's with a one-real-dimensional Kähler moduli “space.” Singularities near the origin will break the Kähler moduli space into two disconnected components, and so as there is no way to smoothly deform the large-radius and Landau-Ginzburg points into one another, Witten indices need not match.

Let us now consider our first example. A degree 5 del Pezzo can be described as $G(2, 5)[1, 1, 1, 1]$. At the Landau-Ginzburg point of the GLSM for $G(2, 5)[1, 1, 1, 1]$, following

³In [26], the dual was described merely as relative Pic^2 – the more explicit description as a $G(2, 5)$ bundle was not given there.

the same analysis as before we find the vanishing locus in \mathbf{P}^3 of 4×4 Pfaffians of a 5×5 skew-symmetric matrix formed from a linear combination of the hyperplane equations.

Note that unlike the Calabi-Yau case, the large-radius and Landau-Ginzburg points no longer describe spaces of the same dimension. The large radius space has complex dimension $6 - 4 = 2$, whereas the Landau-Ginzburg phase is codimension three in \mathbf{P}^3 and hence zero-dimension – a set of points with multiplicity.

4.2.5 A related non-Calabi-Yau example

Consider a GLSM describing the complete intersection of 6 hyperplanes in $G(2, 5)$, *i.e.*, $G(2, 5)[1, 1, 1, 1, 1, 1]$ (or, more compactly, $G(2, 5)[1^6]$). The Landau-Ginzburg point of this GLSM can be determined using previous methods, and is given by the vanishing locus in \mathbf{P}^5 of the 4×4 Pfaffians of a skew-symmetric 5×5 matrix with entries linear in the homogeneous coordinates on \mathbf{P}^5 .

Here, the complete intersection in the Grassmannian $G(2, 5)$ has dimension 0 – it is a set of points – whereas the Pfaffian variety has codimension 3 in \mathbf{P}^5 , hence dimension 2, the opposite of the previous example.

This example is very closely related to the example in the previous section, as the degree 5 del Pezzo $G(2, 5)[1, 1, 1, 1]$ (the large-radius point of the previous example) has an alternate presentation as the vanishing locus in \mathbf{P}^5 of the 4×4 Pfaffians of a skew-symmetric 5×5 matrix with entries linear in the homogeneous coordinates on \mathbf{P}^5 , the Landau-Ginzburg point of the current example. This is a consequence of the equivalence described in section 4.2.2 between $G(2, 5)$ and the vanishing locus in \mathbf{P}^9 of the 4×4 Pfaffians of a generic 5×5 skew-symmetric matrix – by intersecting both sides with four hyperplanes, we obtain the equivalence claimed here.

Not only are the dimension-two phases of this example and the previous one different presentations of the same thing, but so are the dimension-zero phases. Both $G(2, 5)[1^6]$ and the vanishing locus in \mathbf{P}^3 of the 4×4 Pfaffians of a 5×5 skew-symmetric matrix describe the same number of points (including multiplicity).

As a result, we believe the example in this subsection and that in the previous subsection are actually different presentations of the same family of CFT's, in which the interpretation as large-radius and Landau-Ginzburg points has been reversed.

4.2.6 More non-Calabi-Yau examples

Consider a gauged linear sigma model describing the complete intersection of m hyperplanes with $G(2, N)$ for N odd (using the restriction of [4]). Following reasoning that by now should be routine, its Landau-Ginzburg point corresponds to the vanishing locus in \mathbf{P}^{m-1} of the $(N-1) \times (N-1)$ Pfaffians of a skew-symmetric $N \times N$ matrix linear in the homogeneous coordinates on \mathbf{P}^{m-1} , where the $N \times N$ matrix is formed by rewriting a linear combination of the m hyperplanes, with coefficients given by the homogeneous coordinates on \mathbf{P}^{m-1} .

The complete intersection in the Grassmannian $G(2, N)$ has dimension $2(N-2) - m$, whereas the Pfaffian variety has codimension three in \mathbf{P}^{m-1} , hence dimension $m-4$. The canonical bundle of the complete intersection is $\mathcal{O}(m-N)$, whereas following the reasoning in section 4.2.2 the canonical bundle of the Pfaffian variety at the Landau-Ginzburg point is $\mathcal{O}(-m+N)$.

The fact that the canonical bundles of either end have c_1 's of opposite sign gives us another check on these results. After all, the renormalization of the Fayet-Iliopoulos parameter is determined by the gauge linear sigma model and is independent of the phase. If the large-radius phase shrinks under renormalization group flow, *i.e.*, $r \rightarrow 0$, then to be consistent the theory at Landau-Ginzburg must expand to larger values of $|r|$, *i.e.* $r \rightarrow -\infty$, and conversely. A nonlinear sigma model on a positively-curved space will shrink under RG flow, and a nonlinear sigma model on a negatively-curved space will expand under RG flow, so we see that if one phase of the GLSM is positively-curved, then in order for $dr/d\Lambda$ as determined by the GLSM to be consistent across both phases, the other phase must be negatively-curved, and vice-versa.

4.2.7 Interpretation – Kuznetsov’s homological projective duality

In the past, it has been thought that the geometric phases of a gauged linear sigma model were all birational. Here, however, we have seen several examples, both Calabi-Yau and non-Calabi-Yau, of gauged linear sigma models with non-birational phases. One natural question to ask then is, if these phases are not birational, then how precisely are they related mathematically?

In this paper and [9], we would like to propose that these non-birational phases are related by Kuznetsov’s “homological projective duality” [6, 7, 8]. Put another way, here and in [9] we propose that gauged linear sigma models implicitly give a physical realization of Kuznetsov’s homological projective duality.

The physical relevance of Kuznetsov’s work will be discussed in much greater detail in [9], together with more examples (including examples in abelian gauged linear sigma models) and more tests of corner cases.

Very briefly, two spaces are said to be homologically projectively dual if their derived categories of coherent sheaves have pieces in common, in a precise technical sense. Unfortunately, in general there is not a constructive method for producing examples of homological projective duals – given one space, there is not yet an algorithm that will always produce a dual.

However, much is known. For example, it is known that the examples discussed so far are all homologically projectively dual. In other words, the complete intersection of m hyperplanes in $G(2, N)$ for N odd is known to be homologically projectively dual to the vanishing locus in \mathbf{P}^{m-1} of the $(N-1) \times (N-1)$ Pfaffians of a skew-symmetric $N \times N$ matrix linear in the homogeneous coordinates on \mathbf{P}^{m-1} , where the $N \times N$ matrix is formed by rewriting a linear combination of the m hyperplanes. In special cases, this can be understood more systematically. For example, the Grassmannian $G(2, 5)$ itself is homologically projectively dual to a Pfaffian variety of 5×5 skew-symmetric matrices, the space of which lies in $\mathbf{P}(\Lambda^2 \mathbf{C}^5) = \mathbf{P}^9$. Intersecting the Grassmannian with 4 hyperplanes is dual to intersecting the Pfaffian variety with the dual of \mathbf{P}^{9-4} in \mathbf{P}^9 , which is \mathbf{P}^3 .

More generally, we can understand this as follows [33]. Let M denote the dimension of the set of all skew-symmetric $N \times N$ matrices, N odd, choose M generic hyperplanes in $G(2, N)$, and consider the vanishing locus in \mathbf{P}^{M-1} of the $(N-1) \times (N-1)$ Pfaffians of a skew-symmetric $N \times N$ matrix linear in the homogeneous coordinates on \mathbf{P}^{M-1} , where the $N \times N$ matrix is formed by rewriting a linear combination of the M hyperplanes, with coefficients given by the homogeneous coordinates on \mathbf{P}^{M-1} . This variety, call it Y , is homologically projectively dual to $G(2, N)$. If we are given m hyperplanes in $G(2, N)$, they generate $\mathbf{P}^{m-1} \subset \mathbf{P}^{M-1}$, and so the dual is $Y \cap \mathbf{P}^{m-1}$. In the language of section 4.2.2, this duality exchanges orbits $O_2 \leftrightarrow O_{N-3}$, as well as intersections: O_2 intersected with a codimension m subspace A is dual to O_{N-3} intersected with A^\perp , which is a \mathbf{P}^{m-1} . The general result verifies not only the previous example but every example discussed so far, except for the $G(2, 5)$ bundle model in section 4.2.3. For that model, it is believed [34] that the two geometries are homologically projectively dual, though a rigorous proof does not yet exist.

Strictly speaking, what often arises as the duals are not precisely ordinary nonlinear sigma models on spaces, but rather certain resolutions known technically as ‘noncommutative resolutions.’ The term ‘noncommutative’ is a misnomer here, as it does not necessarily imply that there is any noncommutative algebra present, but rather is used as a generic term for anything on which one can do sheaf theory, and ‘noncommutative spaces’ are defined by their sheaves. For example, via matrix factorization one can think of a Landau-Ginzburg model as a ‘noncommutative space’ in this sense. In [9] we will closely examine some examples of homological projective duality in which noncommutative resolutions appear, and we will see how those noncommutative resolutions arise within gauged linear sigma models.

So far we have outlined how non-birational phases of gauged linear sigma models can be

understood as examples of Kuznetsov’s homological projective duality, but we also conjecture the same is true of birational phases also. It is known that some flops between Calabi-Yau’s are examples of homological projective duals, and it is conjectured that all flops are also examples of homological projective duals.

However, although evidence suggests that all phases of gauged linear sigma models might be examples of homological projective duality, the converse may not be true – there are examples of homological projective duals which might not be realizable within gauged linear sigma models. For example, Grassmannians themselves have homological projective duals, but a typical gauged linear sigma model for a Grassmannian has only one phase. Perhaps there are alternate presentations, realizable within gauged linear sigma models, which have multiple phases, or perhaps the duals reflect more subtle aspects of the original gauged linear sigma models.

In the next subsection, we shall consider a possible prediction of Kuznetsov’s work for cases in which the physics is more obscure.

4.2.8 Duals of $G(2, N)$ with N even

The authors of [4] restricted to their analysis of duality in complete intersections in $G(2, N)$ to the case N odd, because the physics in cases for which N is even is both different and more difficult. However, there exist homological projective duals for cases in which N is even, and so we can use homological projective duality to make predictions for such cases.

The duals of N even cases are slightly subtle, so we must proceed carefully. Let us first examine the case of $G(2, 6)[1^6]$ in detail. Naively, the homological projective dual of the complete intersection $G(2, 6)[1^6]$ is a ‘noncommutative resolution’ of a cubic 4-fold, a hypersurface in \mathbf{P}^5 defined by the Pfaffian of a skew-symmetric 6×6 matrix determined by the six hyperplanes in $G(2, 6)$.

Given the match between homological projective duals and phases in previous examples, it is tempting to conjecture that the Landau-Ginzburg phases of $G(2, 6)[1^6]$ might be the Pfaffian variety above. On the other hand, although the complete intersection is Calabi-Yau, the Pfaffian variety is not. It is difficult to understand how a gauged linear sigma model could relate a Calabi-Yau space to a non-Calabi-Yau space, unless the non-Calabi-Yau space has a flux background present so as to preserve supersymmetry.

In fact, the correct answer here is slightly more subtle. The correct dual [6, 28] is a ‘noncommutative⁴ space’ defined by a subcategory of the derived category of the cubic. Two spaces are dual in the relevant mathematical sense if, in essence, their derived categories

⁴As described earlier, but is well worth repeating, the word ‘noncommutative’ is used by mathematicians in this context in a misleading fashion – it does not necessarily indicate the presence of any noncommutativity.

have a piece in common [8][theorem 2.9]. For $G(2, N)[1^N]$ for N odd, the corresponding complete intersection in the Pfaffian variety is already a Calabi-Yau. The derived category of the complete intersection in the Pfaffian variety already matched⁵ that of the Calabi-Yau $G(2, N)[1^N]$ (see *e.g.* [8, 24]), so there was no ‘noncommutative space’ to consider. However, for $G(2, N)[1^N]$ for N even, the derived category of the original space matches only a subcategory of that of the hypersurface in the Pfaffian. So the correct dual in the case of $G(2, 6)[1^6]$ must be something slightly different from the cubic 4-fold described above, something defined by a subcategory of the derived category of the cubic 4-fold. We then have to find a physical theory whose D-branes are described by the pertinent subcategory.

It turns out that there is a very natural candidate for such a physical theory, that should realize the ‘noncommutative space’ defined by a subcategory of the derived category of the cubic 4-fold. In particular, since the cubic is Fano, the category of matrix factorizations of the associated Landau-Ginzburg model is a subcategory [29], and is precisely the subcategory we need to describe. That Landau-Ginzburg model is defined by a cubic superpotential in six variables, which up to finite group factors is a deformation of a Landau-Ginzburg model for an orbifold of a product of two elliptic curves – a K3, in other words. Assuming that the finite group factors work out correctly, and that we have interpreted the structure of that model and its deformations correctly, this finally gives us a physically consistent conjecture for the form of the LG point in $G(2, 6)[1^6]$ – namely, that it is a Landau-Ginzburg model associated to a K3 surface.

So, based on the mathematics of Kuznetsov’s homological projective duality, we propose (as suggested to us by [28]) that the Landau-Ginzburg point of the nonabelian GLSM for $G(2, 6)[1^6]$ is a Landau-Ginzburg model corresponding to a K3 surface.

We will leave to future work a thorough verification that this is indeed the correct description of the Landau-Ginzburg point of this model, although we will mention that it seems very plausible:

- As a consistency check, since $G(2, 6)[1^6]$ is itself a K3, the only other geometries one should find as phases in a GLSM Kähler moduli space are other K3’s, so the result is physically consistent.
- At the level of sheaf theory and D-branes [30, 31], both the derived category of $G(2, 6)[1^6]$ and the category of matrix factorizations of the Landau-Ginzburg model appear [28] as subcategories of the derived category of the cubic fourfold in \mathbf{P}^5 defined as the right orthogonal to $\mathcal{O}(k)$ for $k = -1, -2, -3$, *i.e.*, \mathcal{F} such that $\mathbf{R}\mathrm{Hom}(\mathcal{O}(k), \mathcal{F}) = 0$ for $k = -1, -2, -3$. So the derived categories of the large-radius limit and the proposed interpretation of the Landau-Ginzburg point match on the nose, exactly as needed for

⁵We should be slightly careful. For N odd and $N > 7$, there is a technicality involving the construction of a (noncommutative) resolution of the Pfaffian, so to be precise, these remarks ought to be limited to small N , and considered to be conjectures for larger N .

consistency of the B twist of the gauged linear sigma model.

- Physically, this would mean that the fields fluctuate around a single vacuum at energy cost determined by the Pfaffian of a skew-symmetric 6×6 matrix, which is very plausibly a description of the Landau-Ginzburg point of this gauged linear sigma model.

In addition, we should also mention one other, related, geometry that appears in this context. It is less clear to us what role it plays physically, but for completeness, it should be mentioned. The space of lines in the cubic in \mathbf{P}^5 is another Calabi-Yau – in fact, a hyperKähler 4-fold given by the Hilbert scheme of pairs of points on the K3 $G(2, 6)[1^6]$. Its derived category is nearly (but not quite) identical to that of $G(2, 6)[1^6]$. For more information on this space and its relation to $G(2, 6)[1^6]$ see for example [32].

Similarly, the dual of $G(2, 8)[1^8]$ is a Landau-Ginzburg model defining a ‘noncommutative space’ associated to a ‘noncommutative resolution’ of a degree four hypersurface in \mathbf{P}^7 , where the hypersurface is defined by the Pfaffian of a skew-symmetric 8×8 matrix formed from the eight hyperplanes. The associated Landau-Ginzburg model is defined by a degree four superpotential in eight variables, which is related to an orbifold of the product of two K3 surfaces. In this case we conjecture that the Landau-Ginzburg point of the GLSM for $G(2, 8)[1^8]$ is an orbifold of a Landau-Ginzburg model with a quartic superpotential in eight variables, associated to $(K3 \times K3)/G$ for some G .

Technically [33], we can understand the duals as follows. The complete intersection in a Pfaffian Y dual to $G(2, N)$ for N even is (a certain resolution of) the set of all degenerate skew-symmetric matrices in \mathbf{P}^{M-1} , where M is the dimension of the set of all skew-symmetric $N \times N$ matrices. Such Y can be described as a hypersurface in \mathbf{P}^{M-1} of degree $N/2$, defined by the vanishing locus of the Pfaffian. The dual to a complete intersection of m hyperplanes in $G(2, N)$ is the intersection $Y \cap \mathbf{P}^{m-1}$. Thus, for example, the variety dual to $G(2, 6)[1^6]$ is a degree $6/2 = 3$ hypersurface in $\mathbf{P}^{6-1} = \mathbf{P}^5$.

However, these spaces are never Calabi-Yau, and so can not be the physically-realized homological projective duals to the Calabi-Yau’s $G(2, N)[1^N]$. More generally [33], a Calabi-Yau built as a complete intersection of N hyperplanes in $G(2, N)$ odd will always have a Calabi-Yau homological projective dual; but for N even, the dual variety obtained as a complete intersection in a Pfaffian variety will never be Calabi-Yau (leading us to work with related Landau-Ginzburg models).

Repeating the analysis above, we conjecture that the Landau-Ginzburg points of GLSM’s for $G(2, N)[1^N]$ for N even will always be the ‘correct’ homological projective duals, namely Landau-Ginzburg models defined by degree $N/2$ superpotentials given by the Pfaffian of the skew-symmetric $N \times N$ matrix defined by the hyperplanes. The category of matrix factorizations of these Landau-Ginzburg models will match the derived category of $G(2, N)[1^N]$ for all even N , as needed for consistency of the B twist of the gauged linear sigma model,

and will also be a large distinguished subcategory of the derived category of the pertinent degree $N/2$ hypersurface in \mathbf{P}^{N-1} .

As an aside, the reader might wonder whether $G(2, N)[1^N]$ is hyperKähler for all even N , since our duals above both at least naively appear hyperKähler, and $G(2, N)[1^N]$ always has even complex dimension. However, it can be shown that $G(2, N)[1^N]$ cannot be hyperKähler for $N > 6$. This is because, due to Lefschetz, $h^{2,0}$ of the complete intersection $G(2, N)[1^N]$ matches that of the ambient space $G(2, N)$, which vanishes, but on the other hand, hyperKähler implies holomorphically symplectic which implies $h^{2,0} = 1$.

4.3 Non-birational derived equivalences in abelian GLSMs

Although the example in section 4.2.1 is sometimes advertised as the first example of a gauged linear sigma model involving (a) a Calabi-Yau not presented as a complete intersection, and (b) two non-birationally-equivalent Calabi-Yau's, in fact an abelian gauged linear sigma model with both these properties appeared previously in [5][section 12.2]. There, the gauged linear sigma model described a complete intersection of four degree two hypersurfaces in \mathbf{P}^7 for $r \gg 0$, and was associated with a double cover of \mathbf{P}^3 branched over a degree eight hypersurface (Clemens' octic double solid) for $r \ll 0$. The fact that they are not birational follows from the fact that [35] the complete intersection in \mathbf{P}^7 has no contractible curves, whereas the branched double cover has several ordinary double points. This is another example of Kuznetsov's homological projective duality [7, 9].

The double cover structure at the Landau-Ginzburg limit arises because generically the only massless fields have nonminimal charges. After all, the superpotential

$$W = \sum_i p_i Q_i(x)$$

(where the Q_i are quadric polynomials) can be equivalently rewritten in the form

$$W = \sum_{ij} x_i A^{ij}(p) x_j$$

where A^{ij} is a symmetric matrix with entries linear in the p 's. Away from the locus where A drops rank, *i.e.*, away from the hypersurface $\det A = 0$, the x_i are all massive, leaving only the p_i massless, which all have charge -2 . A GLSM with nonminimal charges describes a gerbe [12, 13, 2], and physically a string on a gerbe is isomorphic to a string on a disjoint union of spaces [5] (see [14] for a short review). Thus, away from the branch locus, we have a double cover, and there is a Berry phase (see [9]) that interweaves the elements of the cover, thus giving precisely the right global structure.

Mathematically, the branched double cover will be singular when the branch locus is singular. (More generally, if the branch locus looks like $f(x_1, \dots, x_n) = 0$, then the double

cover is given by $y^2 = f(x_1, \dots, x_n)$, and it is straightforward to check that the double cover $y^2 = f$ will be smooth precisely where the branch locus $f = 0$ is smooth.) Physically, however, there is no singularity in these models. After all, the F term conditions in this model can be written

$$\begin{aligned}\sum_j A^{ij}(p)x_j &= 0 \\ \sum_{ij} x_i \frac{\partial A^{ij}}{\partial p_k} x_j &= 0\end{aligned}$$

and it can be shown that these equations have no solution – the GLSM is smooth where the geometry is singular. This is one indication that the Landau-Ginzburg point of this GLSM is actually describing a certain (‘noncommutative’) resolution of the branched double cover, matching the prediction of Kuznetsov’s homological projective duality, as will be described in greater detail in [9].

Mathematically, the double cover can be understood as a moduli space of certain bundles on the complete intersection of quadrics [35]. (Each quadric in \mathbf{P}^7 carries two distinct spinor bundles which restrict to bundles on the complete intersection, and when the quadric degenerates, the spinor bundles become isomorphic, hence giving the double cover of \mathbf{P}^3 .)

Also, the twisted derived category of coherent sheaves of the double cover is expected to be [35] isomorphic to the derived category of the complete intersection; such a derived equivalence is to be expected for two geometries on the same GLSM Kähler moduli space.

The primary physical difference between the gauged linear sigma model in [4][section 5], reviewed in section 4.2.1, and [5][section 12.2], reviewed here, is that in the former, at least one ϕ always remains massless (and is removed by quantum corrections), whereas in the latter all of the ϕ are generically massive. Thus, in the latter case one generically has a nonminimally charged field, p , and so gerbes are relevant, whereas in the former there is never a nonminimally-charged-field story.

Also, examples of this form generalize easily. The complete intersection of n quadrics in \mathbf{P}^{2n-1} is related, in the same fashion as above, to a branched double cover of \mathbf{P}^{n-1} , branched over a determinantal hypersurface of degree $2n$. These are Calabi-Yau, for the same reasons as discussed in [5][section 12.2]. (In fact, when $n = 2$, the branched double cover is just the well-known expression of certain elliptic curves as branched double covers of \mathbf{P}^1 , branched over a degree four locus. The fact that K3’s can be described as double covers branched over sextic curves, as realized here for $n = 3$, is described in [36][section 4.5], and the relation between the branched double cover and the complete intersection of quadrics is discussed in [37][p. 145].) This mathematical relationship can be described physically with gauged linear sigma models of essentially the same form as above.

These examples, and Kuznetsov’s homological projective duality, will be discussed in greater detail in [9].

We should also point out that examples of this form indicate that gerbe structures are much more common in GLSM's than previously thought. At minimum, we see that analysis of GLSM's requires understanding the role stacks play in physics. Stacks were originally introduced to physics both to form potentially new string compactifications, and to better understand physical characteristics of string orbifolds, which are described by special cases of strings on stacks. The initial drawbacks included the fact that string compactifications on stacks seemed to suffer from physical inconsistencies, including deformation theory inconsistencies and manifest violations of cluster decomposition, which might be a signal of a potentially fatal presentation dependence. These issues were explored and resolved in [2, 5, 12, 13, 14]. In some sense, this paper and [9] represent a continuation of that work.

5 Quantum cohomology

In this section we collect some physical analyses pertinent to quantum cohomology rings of Grassmannians and flag manifolds.

We begin in subsection 5.1 with a brief review of known results for quantum cohomology rings of such spaces. Quantum cohomology of flag manifolds has been studied in a variety of references; see for example⁶ [3, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49]. In particular, [39] contains a more nearly physics-oriented discussion of nonlinear sigma models.

In subsection 5.2 we briefly outline how those results can be understood in terms of one-loop effective actions in nonabelian gauged linear sigma models.

Finally, in subsection 5.3, we discuss ‘linear sigma model’ moduli spaces of maps from \mathbf{P}^1 into Grassmannians and flag manifolds. These spaces, which are compactifications of spaces of maps, figure prominently in physics discussions of quantum cohomology rings and related invariants. Although they are well-understood for toric varieties, they are less well understood for Grassmannians and flag manifolds. More specifically, there is a natural mathematical notion of what those spaces should be – namely Quot and hyperquot schemes – and those mathematical notions have often been used in mathematical treatments of quantum cohomology rings of Grassmannians and flag manifolds. However, unlike the case of toric varieties, it has not been clear how to get those spaces from thinking about nonabelian gauged linear sigma models, and in fact it has not been completely obvious that the physically relevant ‘linear sigma model’ moduli spaces really are the same as Quot’s and hyperquot’s, rather than some spaces birationally equivalent to them. In the last section we will perform the physical computation of the LSM moduli spaces in several examples, and demonstrate explicitly that physics really does see precisely the same Quot’s and hyperquot’s that are mathematically relevant.

⁶This is merely a sample list of references – it is not intended to be complete or comprehensive.

5.1 Basics

In this subsection we shall briefly review known results for quantum cohomology rings for Grassmannians and flag manifolds.

The quantum cohomology ring of the Grassmannian $G(k, N)$ can be very compactly expressed as follows [3][section 3.2]. Let S, Q be the universal subbundle, quotient bundle as described earlier, then the Chern classes of S generate the cohomology of $G(k, N)$, and the sole modification of quantum corrections is to change the classical constraint⁷

$$c_k(S^\vee)c_{N-k}(Q^\vee) = 0$$

to the constraint

$$c_k(S^\vee)c_{N-k}(Q^\vee) = q \tag{6}$$

It is straightforward to check that the result above reproduces the well-known quantum cohomology ring of a projective space. Recall $\mathbf{P}^{N-1} = G(1, N)$, so in terms of the Grassmannian we have the constraint

$$c_1(S^\vee)c_{N-1}(Q^\vee) = 1$$

Since the tangent bundle is $S^\vee \otimes Q$ and S is a line bundle, we have that $Q^\vee = S^\vee \otimes T^*\mathbf{P}^{N-1}$. Let J denote the degree-two generator of the cohomology ring of \mathbf{P}^{N-1} . Then

$$\begin{aligned} c_{N-1}(Q^\vee) &= c_{N-1}(T^*\mathbf{P}^{N-1}) + c_1(S^\vee)c_{N-2}(T^*\mathbf{P}^{N-1}) + c_1(S^\vee)^2c_{N-3}(T^*\mathbf{P}^{N-1}) + \\ &\quad \dots + c_1(S^\vee)^{N-1} \\ &= (-)^{N-1} \binom{N}{N-1} J^{N-1} + (-)^{N-2} \binom{N}{N-2} J^{N-1} + \dots + \binom{N}{0} J^{N-1} \\ &= (-)^{N-1} J^{N-1} \end{aligned}$$

where we have used the fact that $S^\vee \cong \mathcal{O}(1)$ and $c_i(T^*\mathbf{P}^{N-1}) = (-)^i \binom{N}{i}$. Thus, the quantum relation

$$c_k(S^\vee)c_{N-k}(Q^\vee) = q$$

reduces in the special case $k = 1$ to the relation

$$J^N = (-)^{N-1}q$$

which up to irrelevant signs is the quantum cohomology relation on \mathbf{P}^{N-1} .

⁷Given the short exact sequence relating S and Q , the total Chern classes are necessarily related by $c(S^\vee)c(Q^\vee) = 1$. The first $N - k$ cohomology classes in the expansion can be used to solve for the Chern classes of Q^\vee in terms of those of S^\vee ; the remaining ones are relations among the cohomology classes, including the one below at the last step.

For the flag manifold $F(k_1, \dots, k_n, N)$, the quantum cohomology is somewhat more complicated to express [39][section 5]. The classical cohomology ring is generated by the Chern classes of the bundles S_i/S_{i-1} . Let $x_j^{(i)}$ denote the j th Chern class of S_i/S_{i-1} (in conventions where $S_0 = 0$ and $S_{n+1} = \mathcal{O}^{\oplus N}$), then the classical cohomology of the flag manifold can be expressed as the ring

$$\mathbf{C}[x_1^{(1)}, \dots, x_{k_1}^{(1)}, x_1^{(2)}, \dots, x_{N-k_n}^{(n+1)}]$$

modulo the ideal generated by the coefficients of the polynomial

$$P(\lambda) = \lambda^N - \prod_{i=1}^{n+1} \left(\lambda^{k_i - k_{i-1}} + \lambda^{k_i - k_{i-1} - 1} x_1^{(i)} + \dots + x_{k_i - k_{i-1}}^{(i)} \right)$$

Let us check that this reproduces the classical cohomology ring of the Grassmannian $G(k, N)$. Following the procedure above, the cohomology is given by the ring

$$\mathbf{C}[x_1^{(1)}, \dots, x_k^{(1)}, x_1^{(2)}, \dots, x_{N-k}^{(2)}]$$

modulo the ideal generated by coefficients of the polynomial above, which in this case becomes

$$\lambda^N - \left(\lambda^k + \lambda^{k-1} x_1^{(1)} + \dots + x_k^{(1)} \right) \left(\lambda^{N-k} + \lambda^{N-k-1} x_1^{(2)} + \dots + x_{N-k}^{(2)} \right) \quad (7)$$

Multiplying this out, we find that the coefficient of λ^N is 0, the coefficient of λ^{N-1} is

$$x_1^{(1)} + x_1^{(2)}$$

the coefficient of λ^{N-2} is

$$x_2^{(1)} + x_1^{(1)} x_1^{(2)} + x_2^{(2)}$$

and so forth. Now, let us compare to the Grassmannian. The $x_j^{(i)}$ are the Chern classes of the universal subbundle S and quotient bundle Q :

$$x_i^{(1)} = c_i(S), \quad x_i^{(2)} = c_i(V/S) = c_i(Q)$$

Because S and Q are related by the short exact sequence

$$0 \longrightarrow S \longrightarrow V \longrightarrow Q \longrightarrow 0$$

we have that $c(S)c(Q) = 1$, where c denotes the total Chern class. But

$$c(S)c(Q) = 1 + (c_1(S) + c_1(Q)) + (c_2(S) + c_1(S)c_1(Q) + c_2(Q)) + \dots$$

and so we see the coefficients of $c(S)c(Q)$ at any given order in cohomology are the same as the coefficients of the polynomial $P(\lambda)$ above. Thus, we recover the classical cohomology ring of the Grassmannian as a special case of the classical cohomology ring of the flag manifold above.

The quantum cohomology of the flag manifold is obtained by deforming the relations as follows [39]. Introduce complex parameters q_1, \dots, q_n , and instead of taking the relations to be defined by the coefficients of λ in the polynomial $P(\lambda)$ above, take them to be the coefficients of λ in the polynomial

$$\lambda^N - \det(A + \lambda I)$$

where A is the $N \times N$ matrix defined by

$$\begin{pmatrix} x_1^{(1)} & \cdots & x_{k_1}^{(1)} & 0 & \cdots & -(-)^{k_2-k_1}q_1 & \cdots & 0 & \cdots & \cdots & 0 & 0 \\ -1 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & -1 & x_1^{(2)} & \cdots & x_{k_2-k_1}^{(2)} & \cdots & -(-)^{k_3-k_2}q_2 & \cdots & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & \cdots & x_1^{(n+1)} & \cdots & x_{N-k_n-1}^{(n+1)} & x_{N-k_n}^{(n+1)} \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & \cdots & -1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & \cdots & 0 & \cdots & -1 & 0 \end{pmatrix}$$

In the special case of the Grassmannian $G(k, N)$, the matrix $A + \lambda I$ is given by

$$\begin{pmatrix} x_1^{(1)} + \lambda & x_2^{(1)} & \cdots & x_{k-1}^{(1)} & x_k^{(1)} & 0 & 0 & \cdots & 0 & -(-)^{N-k}q \\ -1 & \lambda & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \ddots & \lambda & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & -1 & \lambda & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & -1 & x_1^{(2)} + \lambda & x_2^{(2)} & \cdots & x_{N-k-1}^{(2)} & x_{N-k}^{(2)} \\ 0 & 0 & \cdots & 0 & 0 & -1 & \lambda & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & \lambda & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & -1 & \lambda \end{pmatrix}$$

It can be shown that

$$\det(A + \lambda I) = \left(\lambda^k + \lambda^{k-1}x_1^{(1)} + \cdots + x_k^{(1)} \right) \left(\lambda^{N-k} + \lambda^{N-k-1}x_1^{(2)} + \cdots + x_{N-k}^{(2)} \right) + (-)^{N-k-1}q$$

The relations in the polynomial ring are the coefficients of

$$\lambda^N - \det(A + \lambda I)$$

Comparing to equation (7) we see that the relations for the quantum cohomology ring are nearly identical to those for the classical cohomology ring, except for the one modification

$$x_k^{(1)} x_{N-k}^{(2)} + (-)^{N-k-1} q = 0$$

which the reader will recognize as the quantum cohomology relation for the Grassmannian described earlier in equation (6), up to an irrelevant sign.

For the flag manifold $F(k_1, k_2, N)$, it can be shown that

$$\det(A + \lambda I) = Q_1 Q_2 Q_3 + (-)^{N-k_2-1} q_2 Q_1 - (-)^{k_2-k_1-1} q_1 Q_3$$

where

$$\begin{aligned} Q_1 &= \lambda^{k_1} + \lambda^{k_1-1} x_1^{(1)} + \cdots + x_{k_1}^{(1)} \\ Q_2 &= \lambda^{k_2-k_1} + \lambda^{k_2-k_1-1} x_1^{(2)} + \cdots + x_{k_2-k_1}^{(2)} \\ Q_3 &= \lambda^{N-k_2} + \lambda^{N-k_2-1} x_1^{(3)} + \cdots + x_{N-k_2}^{(3)} \end{aligned}$$

As before, when $q_1 = q_2 = 0$, the quantum cohomology relations one derives from the above reduce to the classical cohomology relations. For cohomology degrees lower than both k_2 and $N - k_1$, the relations one derives from the above are identical to the classical relations. However, in lower degrees there are quantum corrections. From the coefficient of λ^{N-k_2} we get the relation

$$\begin{aligned} &x_{k_1}^{(1)} x_{k_2-k_1}^{(2)} + x_{k_1-1}^{(1)} x_{k_2-k_1}^{(2)} x_1^{(3)} + x_{k_1}^{(1)} x_{k_2-k_1-1}^{(2)} x_1^{(3)} + \cdots \\ &+ (-)^{k_2-k_1} q_1 + (-)^{N-k_2-1} q_2 x_{k_1+k_2-N}^{(1)} = 0 \end{aligned}$$

From the coefficient of λ^{N-k_2-1} we get the relation

$$\begin{aligned} &x_{k_1}^{(1)} x_{k_2-k_1}^{(2)} x_1^{(3)} + x_{k_1-1}^{(1)} x_{k_2-k_1}^{(2)} x_2^{(3)} + x_{k_1}^{(1)} x_{k_2-k_1-1}^{(2)} x_2^{(3)} + \cdots \\ &+ (-)^{k_2-k_1} q_1 x_1^{(3)} + (-)^{N-k_2-1} q_2 x_{k_1+k_2+1-N}^{(1)} = 0 \end{aligned}$$

From the coefficient of λ^{k_1} we get the relation

$$\begin{aligned} &x_{k_2-k_1}^{(2)} x_{N-k_2}^{(3)} + x_1^{(1)} x_{k_2-k_1-1}^{(2)} x_{N-k_2}^{(3)} + x_1^{(1)} x_{k_2-k_1}^{(2)} x_{N-k_2-1}^{(3)} + \cdots \\ &+ (-)^{N-k_2-1} q_2 + (-)^{k_2-k_1} q_1 x_{N-k_2-k_1}^{(3)} = 0 \end{aligned}$$

and so forth. In each case, q_1 acts like a cohomology class of degree k_2 , and q_2 acts like a cohomology class of degree $N - k_1$.

5.2 One-loop effective action arguments

In [50], Batyrev's conjecture for quantum cohomology rings of toric varieties was derived physically from the one-loop effective action for the two-dimensional gauged linear sigma

model. Those techniques were extended to toric stacks in [2], and in this subsection we shall briefly outline some heuristics showing how analogous results arise in gauged linear sigma models for flag manifolds. We will not claim to have a derivation, but rather merely hope to shed some insight into how the quantum cohomology ring of a flag manifold arises at the level of nonabelian gauged linear sigma models.

First, recall from [3][section 4.2] that in the gauged linear sigma model for a Grassmannian, the field σ is identified in the low-energy theory with the curvature form for the $U(k)$ gauge symmetry, which mathematically generates the cohomology of $G(k, N)$. One way to see this identification of σ involves the equations of motion for σ at low energies. The relevant part of the action is

$$\bar{\phi}_{is}\{\sigma, \bar{\sigma}\}_j^i \phi^{js} + \sqrt{2}\bar{\psi}_{+is}\bar{\sigma}_j^i \psi_-^{js} + c.c.$$

At low energies the field ϕ^{is} have a vacuum expectation value, and plugging that in and computing equations of motion yields

$$\sigma_j^i \propto \frac{1}{r} \sum_s \bar{\psi}_{+js} \psi_-^{is}$$

The interpretation of σ in the low-energy theory follows from the fact that the ψ 's are tangent to the Grassmannian in that theory.

Now, using that interpretation of σ , let us give a very brief and heuristic outline of how to apply the methods of [50] to the present case. The superpotential in the one-loop effective action for the Grassmannian was calculated in [4][equation (2.16)] along the Coulomb branch, and has the form

$$W = -t \sum_{a=1}^k \Sigma_a - \sum_{a=1}^k N \Sigma_a (\log \Sigma_a - 1).$$

The fields Σ_a are superfields whose lowest components are eigenvalues of σ . The vacua have the form $\Sigma_a^N = \exp(-t)$ for each a . Since σ is the curvature of the universal subbundle S , the Σ_a should be interpreted as Chern roots of the universal subbundle. Naively, the statement above looks as if it might be interpreted as a quantum correction to $c_N(S)$, but no such Chern class exists for S . Instead, using the fact that the Chern classes of the universal quotient bundle Q are determined by those of S , a more sensible interpretation would be as a quantum correction to $c_k(S)c_{N-k}(Q)$, which is the form of the quantum cohomology of the Grassmannian.

Again, this is not meant by any stretch to be a thorough derivation, but rather is only intended to be a brief heuristic argument linking the quantum cohomology of the Grassmannian and the one-loop effective action, in the spirit of [50].

A similar heuristic argument can be applied in the case of more general flag manifolds. Here, the interpretation of the σ fields is more interesting. Let us apply the same arguments as above to a σ field coupling to a $U(k_i)$ factor in the gauge group of a gauged linear sigma

model describing $F(k_1, \dots, k_n, N)$. Relevant terms in the action are

$$-\bar{\phi}_{is}\{\sigma, \bar{\sigma}\}_j^i \phi^{js} + \bar{\phi}_{ia}\{\sigma, \bar{\sigma}\}_j^i \tilde{\phi}^{ja} - \sqrt{2}\bar{\psi}_{+is}\bar{\sigma}_j^i \psi_-^{js} + c.c. + \sqrt{2}\bar{\psi}_{+ia}\bar{\sigma}_j^i \tilde{\psi}_-^{ja} + c.c.$$

where ϕ^{is} , ψ^{is} are bifundamentals in the $(\mathbf{k}_{i-1}, \bar{\mathbf{k}}_i)$ representation of $U(k_{i-1}) \times U(k_i)$ and $\tilde{\phi}^{ia}$, $\tilde{\psi}^{ia}$ are bifundamentals in the $(\mathbf{k}_i, \bar{\mathbf{k}}_{i+1})$ representation of $U(k_i) \times U(k_{i+1})$. Plugging in vacuum expectation values for the ϕ , $\tilde{\phi}$, and applying D-terms, we find that we find that in the low-energy theory the equations of motion for σ have the form

$$\sigma_j^i \propto -\frac{1}{r_i} \sum_s \bar{\psi}_{+js} \psi_-^{is} + \frac{1}{r_i} \sum_a \bar{\psi}_{+ja} \tilde{\psi}_-^{ia} \quad (8)$$

Instead of being the curvature of one universal subbundle S_i , here the field σ should now have the interpretation of being the difference in curvatures between two subbundles. Indeed, if we proceed as in [3] and interpret the $\bar{\psi}_{+js} \psi_-^{is}$ term as the curvature of S_{i-1} , and the $\bar{\psi}_{+ja} \tilde{\psi}_-^{ia}$ term as the curvature of S_i , then we should interpret σ as the curvature of S_i/S_{i-1} .

Now, let us compute the one-loop effective action for the flag manifold $F(k_1, \dots, k_n, N)$. As in [4], we shall work along the Coulomb branch in a region where every $U(k_i)$ gauge symmetry has been Higgsed down to $U(1)^{k_i}$. In general terms, from [50][equ'n (3.36)], the one-loop effective superpotential in an abelian gauge theory with charged matter should have the form

$$\tilde{W} = \sum_a \Sigma_a \left[-t_a - \sum_i Q_i^a \left(\log \left(\sum_b Q_i^b \Sigma_b \right) - 1 \right) \right]$$

In the case of the Grassmannian $G(k, N)$, there were N fundamentals of $U(k)$, which became N selectrons of charge 1 under each of the $U(1)$'s in $U(1)^k$. In the case of the flag manifold $F(k_1, \dots, k_n, N)$, we have bifundamentals $(\mathbf{k}_i, \bar{\mathbf{k}}_{i+1})$ which become a matrix of selectrons. The ij th entry in the matrix $\mathbf{C}^{k_i k_{i+1}}$ has charge +1 under the i th $U(1)$ in $U(1)^{k_i}$, and charge -1 under the j th $U(1)$ in $U(1)^{k_{i+1}}$. Putting this together, we get that the effective superpotential should be given by

$$\tilde{W} = \sum_{i=1}^n \sum_{a=1}^{k_i} \Sigma_{ia} \left[-t_i - \sum_{s=1}^{k_{i+1}} \left(\log(\Sigma_{ia} - \Sigma_{(i+1)s}) - 1 \right) + \sum_{s=1}^{k_{i-1}} \left(\log(\Sigma_{(i-1)s} - \Sigma_{ia}) - 1 \right) \right]$$

with obvious modifications at the endpoints.

For example, in the special case $F(k_1, k_2, N)$, the effective superpotential is given by

$$\begin{aligned} \tilde{W} = & \sum_{a=1}^{k_1} \Sigma_{1a} \left[-t_1 - \sum_{s=1}^{k_2} \left(\log(\Sigma_{1a} - \Sigma_{2s}) - 1 \right) \right] \\ & + \sum_{a=1}^{k_2} \Sigma_{2a} \left[-t_2 - \sum_{s=1}^N \left(\log(\Sigma_{2a}) - 1 \right) + \sum_{s=1}^{k_1} \left[\log(\Sigma_{1s} - \Sigma_{2a}) - 1 \right] \right] \end{aligned}$$

The equations of motion for Σ_{1a} are given by

$$\prod_{s=1}^{k_2} (\Sigma_{1a} - \Sigma_{2s}) = \exp(-t_1) \equiv q_1 \quad (9)$$

The equations of motion for Σ_{2a} are given by

$$\begin{aligned} \Sigma_{2a}^N &= \exp(-t_2) \prod_{s=1}^{k_1} (\Sigma_{1s} - \Sigma_{2a}) \\ &= q_2 \prod_{s=1}^{k_1} (\Sigma_{1s} - \Sigma_{2a}) \end{aligned}$$

where $q_i \equiv \exp(-t_i)$.

Now, let us compare to the quantum cohomology ring relations for $F(k_1, k_2, N)$ computed earlier. We found, for example,

$$\begin{aligned} x_{k_1}^{(1)} x_{k_2-k_1}^{(2)} + x_{k_1-1}^{(1)} x_{k_2-k_1}^{(2)} x_1^{(3)} + x_{k_1}^{(1)} x_{k_2-k_1-1}^{(2)} x_1^{(3)} + \dots \\ + (-)^{k_2-k_1} q_1 + (-)^{N-k_2-1} q_2 x_{k_1+k_2-N}^{(1)} = 0 \end{aligned}$$

and

$$\begin{aligned} x_{k_2-k_1}^{(2)} x_{N-k_2}^{(3)} + x_1^{(1)} x_{k_2-k_1-1}^{(2)} x_{N-k_2}^{(3)} + x_1^{(1)} x_{k_2-k_1}^{(2)} x_{N-k_2-1}^{(3)} + \dots \\ + (-)^{N-k_2-1} q_2 + (-)^{k_2-k_1} q_1 x_{N-k_2-k_1}^{(3)} = 0 \end{aligned}$$

where $x_j^{(i)} = c_j(S_i/S_{i-1})$. The non-quantum-corrected ring relations allow us to solve for the $x_j^{(3)}$ in terms of the $x_j^{(1)}$ and $x_j^{(2)}$. Note that in these polynomial equations, q_1 appears as if it were a cohomology class of degree k_2 , and q_2 appears as if it were a cohomology class of degree $N - k_1$.

Now, from equation (8), we identify Σ_{1a} with the Chern roots of S_1 and Σ_{2a} with the Chern roots of S_2/S_1 , *i.e.* the $x_j^{(1)}$ are built from the Σ_{1a} and the $x_j^{(2)}$ are built from the Σ_{2a} . In both cases, we find that the q_i are related to polynomials in the Chern classes – obtained by expanding out the products $\prod(\Sigma_{1a} - \Sigma_{2s})$ in the equations of motion for Σ_{1a} and Σ_{2a} . Furthermore, and more importantly, note that those same equations of motion are consistent with q_1 behaving as a class of degree k_2 and q_2 behaving as a class of degree $N - k_1$.

As our purpose in this section was merely to heuristically outline how the quantum cohomology ring arises in nonabelian gauged linear sigma models, we shall not pursue this direction further.

5.3 Linear sigma model moduli spaces

In this subsection we shall discuss linear sigma model moduli spaces for Grassmannians and flag manifolds, which are relevant to some computations of quantum cohomology rings. Such

moduli spaces are well-understood physically for abelian gauged linear sigma models, but to our knowledge have not been worked out for nonabelian gauged linear sigma models, where the story is necessarily more complicated. There is a natural mathematical proposal for such moduli spaces – namely that they should be Quot schemes and hyperquot schemes – and we shall see explicitly in examples that the physical moduli spaces really are precisely the appropriate Quot and hyperquot schemes, as one would have naively guessed from mathematics, and not some other related (possibly birational) spaces instead.

Before doing so, however, let us first review some basic facts regarding honest maps

$$\mathbf{P}^1 \longrightarrow F(k_1, \dots, k_n, N)$$

Such a map corresponds to a flag of bundles over \mathbf{P}^1 :

$$\mathcal{E}_1 \hookrightarrow \mathcal{E}_2 \hookrightarrow \dots \hookrightarrow \mathcal{E}_n \hookrightarrow \mathcal{O}_{\mathbf{P}^1}^{\oplus N}$$

where $\text{rank } \mathcal{E}_i = k_i$. Put simply, this is because a map from \mathbf{P}^1 into the flag manifold is given by specifying, for each point on \mathbf{P}^1 , a flag, which defines a point on the flag manifold. Those flags fit together to form a flag of bundles on \mathbf{P}^1 . The multidegree of the map is defined by the degrees of the bundles \mathcal{E}_i above.

5.3.1 Grassmannians

What is the linear sigma model moduli space for target $G(k, N)$?

Recall the linear sigma model describes a GIT quotient of \mathbf{C}^{kN} by $GL(k)$. The kN chiral superfields ϕ_{is} are naturally N copies of the fundamental representation of $GL(k)$. The zero modes of the ϕ_{is} are given as elements of

$$H^0(\mathbf{P}^1, \phi^*(S^\vee \otimes V))$$

In a nonlinear sigma model on a Grassmannian, to specify the degree of the map we would need merely specify an integer, ultimately because $H^2(G(k, N), \mathbf{Z}) = \mathbf{Z}$. In a linear sigma model, however, we must specify a partition of d into k integers a_i . These integers specify the splitting type on the \mathbf{P}^1 worldsheet:

$$\phi^* S^\vee = \oplus_i \mathcal{O}(a_i)$$

where

$$\sum_i a_i = d$$

In principle, all splitting types should contribute physically, but the gauged linear sigma model appears to provide a stratification of the full moduli space.

From Hirzebruch-Riemann-Roch,

$$\begin{aligned}\chi(S^\vee \otimes V) &= c_1(S^\vee \otimes V) + kN(1 - g) \\ &= Nd + Nk \\ &= N(k + d)\end{aligned}$$

We can expand out the zero modes of each chiral superfield in the usual fashion. Note that the original $U(k)$ symmetry is broken by the choice of a_i to some subset. Only when all the a_i are the same is the complete $U(k)$ preserved.

Also note that we can recover an ordinary projective space in the case that $k = 1$, in which case the subtlety above does not arise.

We shall consider several examples next, and study the various strata appearing in the gauged linear sigma model moduli space. Then, after examining several examples, we will discuss how the complete moduli spaces can be understood, as Quot schemes and hyperquot schemes.

5.3.2 Balanced strata example

First, consider degree km maps from \mathbf{P}^1 into $G(k, N)$, so that $km = d$, and use a partition $a_i = m$ for all i . In this case, the entire $U(k)$ symmetry will be preserved. Each chiral superfield can be expanded

$$\phi_{is} = \phi_{is}^0 u^m + \phi_{is}^1 u^{m-1} v + \cdots \phi_{is}^m v^m$$

where u, v are homogeneous coordinates on the worldsheet. The coefficients ϕ_{is}^a transform in the fundamental representation of $U(k)$, which suggests that the linear sigma model moduli space is given by

$$\mathbf{C}^{(m+1)kN} // GL(k)$$

One example of such a quotient, depending upon the excluded set, is the Grassmannian $G(k, N(m+1))$, but we shall see that this stratum is not precisely a Grassmannian.

To completely determine this stratum, we need to specify the excluded set. In order for the LSM moduli space to be given by the Grassmannian $G(k, N(m+1))$, we would need that for generic u, v , the k vectors $(\phi_i)_s^a = \phi_{is}^a$ in $\mathbf{C}^{N(m+1)}$ are orthonormal. Instead we have a slightly different condition. The D-terms in the original LSM say that the ϕ_{is} should form k orthonormal vectors in \mathbf{C}^N . Applying this constraint guarantees that for generic points on the moduli space, we will be describing a flag of bundles on \mathbf{P}^1 , which as discussed earlier describes a map into a flag manifold. In terms of zero modes, this implies that for generic

$u, v,$

$$\begin{aligned}
& \sum_s \left(\phi_{is}^0 u^m + \phi_{is}^1 u^{m-1} v + \dots \right) \left(\phi_{js}^0 u^m + \phi_{js}^1 u^{m-1} v + \dots \right) \\
&= \sum_s \phi_{is}^0 \phi_{js}^0 u^{2m} + \sum_s \left(\phi_{is}^0 \phi_{js}^1 + \phi_{is}^1 \phi_{js}^0 \right) u^{2m-1} v + \dots \\
&= \delta_{ij} r
\end{aligned}$$

More compactly, we can express this condition by saying that over every point on \mathbf{P}^1 , the $k \times N$ matrix ϕ_{is} must be of rank k . To be in the Grassmannian $G(k, N(m+1))$, by contrast, a weaker condition need be satisfied, namely that the k vectors in $\mathbf{C}^{N(m+1)}$ with coordinates $(\phi_i)_s^a = \phi_{is}^a$, must be linearly independent. To see that our condition is stronger, consider the following example. Take any k and $N, m_k - 2$. If the ϕ_{is}^a are defined by, for $s = 1$, a degree m polynomial with k different terms and, for $s > 1$, identically zero degree m polynomials, the result is a perfectly good point of the Grassmannian but which does not satisfy our stronger condition, as the matrix ϕ_{is} is of rank 1 not rank k .

So, briefly, this component of the moduli space has the form $U/GL(k)$ for U an open subset of $\mathbf{C}^{kN(m+1)}$, and is a proper subset of the Grassmannian $G(k, N(m+1))$.

In the special case that $k = 1$, this simplifies slightly, and the condition to be on the LSM moduli space becomes identical to that for the Grassmannian, so the moduli space becomes $G(1, N(m+1)) = \mathbf{P}^{N(m+1)-1}$, reproducing the standard result for linear sigma model moduli spaces of maps $\mathbf{P}^1 \rightarrow \mathbf{P}^{N-1}$.

5.3.3 Degree 1 maps example

Next, let us consider degree one maps from \mathbf{P}^1 into $G(2, N)$. We can partition as $(1, 0)$ or $(0, 1)$. Physics only sees unordered partitions, so, there is a single stratum. Here we run into a subtlety: since the line bundles in the partition are no longer symmetric, we can not quotient precisely by $GL(2)$.

One naive guess would be to restrict to a subgroup of $GL(2)$ that preserves the stratification, such as $(\mathbf{C}^\times)^2$. One would get that the pieces of the moduli space look like

$$(\mathbf{C}^N // GL(1)) \times (\mathbf{C}^{2N} // GL(1)) = \mathbf{P}^{N-1} \times \mathbf{P}^{2N-1}$$

However, that naive guess is not quite right. Not only is the truncation of $GL(2)$ to $(\mathbf{C}^\times)^2$ rather unnatural, but it also gives pieces of the wrong dimension. These moduli spaces should be some sort of compactification of spaces of maps into Grassmannians, and the space of maps of degree d from \mathbf{P}^1 into the Grassmannian $G(k, N)$ has [48][theorem 2.1] dimension $k(N-k) + dN$. Here, we are considering degree 1 maps from \mathbf{P}^1 into $G(2, N)$, and so our space should have dimension $2(N-2) + N = 3N - 4$. Instead, the naive guess above

has dimension $3N - 2$, which is too big – by exactly the same amount as the dimension of that part of the group $GL(2)$ that we truncated.

The correct computation is somewhat more complicated. The correct symmetry group is the group of global automorphisms of the vector bundle $\mathcal{O} \oplus \mathcal{O}(1)$. This is not a subgroup of $GL(2)$. Suppose more generally we were describing degree $p + q$ maps into $G(2, N)$, and considering a stratum defined by the splitting $\mathcal{O}(p) \oplus \mathcal{O}(q)$. Infinitesimal automorphisms will be defined by a matrix whose entries are of the form

$$\left[\begin{array}{cc} H^0(\mathcal{O}(p)^\vee \otimes \mathcal{O}(p)) = H^0(\mathcal{O}) & H^0(\mathcal{O}(p)^\vee \otimes \mathcal{O}(q)) = H^0(\mathcal{O}(q - p)) \\ H^0(\mathcal{O}(q)^\vee \otimes \mathcal{O}(p)) = H^0(\mathcal{O}(p - q)) & H^0(\mathcal{O}(q)^\vee \otimes \mathcal{O}(q)) = H^0(\mathcal{O}) \end{array} \right]$$

In the present example, corresponding to the splitting $\mathcal{O} \oplus \mathcal{O}(1)$, this reduces to

$$\left[\begin{array}{cc} H^0(\mathcal{O}) & H^0(\mathcal{O}(-1)) = 0 \\ H^0(\mathcal{O}(1)) = \mathbf{C}^2 & H^0(\mathcal{O}) \end{array} \right]$$

which is the Lie algebra of some non-reductive group that is an *extension* of $(\mathbf{C}^\times)^2$. So we should not be quotienting $(\mathbf{C}^N \times \mathbf{C}^{2N})$ by $(\mathbf{C}^\times)^2$, but rather by a group G which has dimension two larger. As a result, the correct moduli space in this example should have dimension $3N - 4$ instead of $3N - 2$, which matches expectations.

Note that the off-diagonal \mathbf{C}^2 acts on the space $\mathbf{C}^N \times \mathbf{C}^{2N}$, *i.e.*, $1 \times N$ and $2 \times N$ matrices, by leaving the $1 \times N$ matrices invariant, and subtracts from the $2 \times N$ matrices another $2 \times N$ matrix formed by tensoring a two-vector of \mathbf{C}^2 with the $1 \times N$ matrix.

As a quick check, consider the case of a splitting $\mathcal{O}(p) \oplus \mathcal{O}(q)$. In this case, one would expect that one should quotient by $GL(2)$, as we did previously. From the general story above, the Lie algebra of the correct group has the form

$$\left[\begin{array}{cc} H^0(\mathcal{O}) = \mathbf{C} & H^0(\mathcal{O}(p - p)) = H^0(\mathcal{O}) = \mathbf{C} \\ H^0(\mathcal{O}(p - p)) = H^0(\mathcal{O}) = \mathbf{C} & H^0(\mathcal{O}) = \mathbf{C} \end{array} \right]$$

i.e. 2×2 matrices, exactly matching the Lie algebra of $GL(2)$.

We can also use the quantum cohomology of the Grassmannian described earlier to get a quick consistency check of the dimension of this moduli space. Recall that in the Grassmannian $G(k, N)$, the quantum cohomology ring is defined by

$$c_k(S^\vee) c_{N-k}(Q^\vee) = q$$

Roughly, this OPE is realized by a correlation function of the form

$$\langle (\bar{\psi}\psi)^{k(N-k)} (\bar{\psi}\psi)^N \rangle = q$$

where the $\overline{\psi}\psi$'s correspond to two-forms on the moduli space, and the correlation function will be nonzero for maps of fixed degree when the sum⁸ of the degrees equals the dimension of the moduli space. Since the moduli space here has dimension $3N - 4$, which is the same as $k(N - k) + N$ for $k = 2$, we see that the dimension of the moduli space we just computed is consistent with the quantum cohomology ring.

5.3.4 Degree two maps example

As another example, consider degree two maps from \mathbf{P}^1 into $G(2, N)$. Here, the relevant partitions are $(2, 0)$, $(1, 1)$, $(0, 2)$. As physics only sees unordered partitions, there are two strata, corresponding to $(2, 0)$ and $(1, 1)$.

The partition $(1, 1)$ gives rise to a stratum that is a large open subset of $\mathbf{C}^{4N} // GL(2) = G(2, 2N)$.

The partition $(2, 0)$ gives rise to the stratum

$$(\mathbf{C}^{3N} \times \mathbf{C}^N) // G$$

where G is an extension of $(\mathbf{C}^\times)^2$, as discussed previously.

The partitions fit together in the following form: in the rank 2 case, if $a < b$, the closure of the partition corresponding to $\mathcal{O}(a) \oplus \mathcal{O}(b)$ contains the partition for $\mathcal{O}(a - 1) \oplus \mathcal{O}(b + 1)$. (A quick way to see that this is in the closure is to consider the example

$$\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(1) \longrightarrow \mathcal{O}(1)$$

of maps between bundles on \mathbf{P}^1 . Letting homogeneous coordinates on \mathbf{P}^1 be denoted $[x, y]$, take the map above to be defined by the triple (x, y, ϵ) for some constant ϵ . For $\epsilon \neq 0$, the kernel of the bundle map above is \mathcal{O}^2 , but for $\epsilon = 0$, the kernel is $\mathcal{O}(-1) \oplus \mathcal{O}(1)$. Thus, $\mathcal{O}(-1) \oplus \mathcal{O}(1)$ lies in the closure of \mathcal{O}^2 .) Roughly speaking, when the bundle is very ‘balanced,’ in the obvious sense, it is general, whereas when it is very unbalanced, it is very special.

As in the last section, we can get a quick consistency check of the dimension of this moduli space from thinking about the quantum cohomology ring. As described earlier, the OPE structure is defined by the relation

$$c_k(S^\vee) c_{N-k}(Q^\vee) = q$$

In the present case, for degree two maps, this OPE structure is consistent with a correlation function of the general form

$$< (\overline{\psi}\psi)^{k(N-k)} (\overline{\psi}\psi)^{2(k+(N-k))} > = q^2$$

⁸In general, this statement would be modified by Euler classes of obstruction bundles, but in this simple case there is no obstruction bundle, so this reduces to dimension counting.

where we are using the fact that there are no Euler classes of obstruction bundles in the computation. In order for the correlation function above to be nonzero in this instanton sector, the moduli space of maps must have dimension $k(N - k) + 2N$, and indeed, the generic stratum of our moduli space has dimension $4N - 4 = 2(N - 2) + 2N$, as desired for $k = 2$.

Note that the nongeneric stratum, corresponding to the partition $(2, 0)$, lies in the closure of the stratum corresponding to that for the partition $(1, 1)$ and so must have a smaller dimension. This dimension is seen to be $4N - 5$, *i.e.* this stratum has codimension 1 in the full moduli space. This follows immediately from examination of the group G , which for this stratum has dimension 5, strictly bigger than the dimension of $GL(2)$.

5.3.5 Interpretation – Quot and hyperquot schemes

In abelian GLSM's, the entire LSM moduli space could be constructed and analyzed at a single stroke, but we have seen that in nonabelian GLSM's, we only immediately compute strata of the full LSM moduli space. One is naturally led to ask, what is the complete space of which these are strata?

We propose that the answer is that the complete LSM moduli spaces in nonabelian GLSM's for Grassmannians are spaces known as Quot schemes.

Quot schemes generalize both Grassmannians and Hilbert schemes. They parametrize quotients of bundles. In the present case, a (linear sigma model) moduli space of degree d maps from \mathbf{P}^1 into $G(k, N)$ is given by a Quot scheme describing rank k bundles on \mathbf{P}^1 together with an embedding into \mathcal{O}^N , or alternatively quotients of \mathcal{O}^N bundles. In the special case of maps $\mathbf{P}^1 \rightarrow G(1, N) = \mathbf{P}^{N-1}$, the Quot scheme is the projective space $\mathbf{P}^{N(d+1)-1}$, reproducing the standard result for linear sigma model moduli spaces of maps $\mathbf{P}^1 \rightarrow \mathbf{P}^{N-1}$.

Quot schemes describing bundles on \mathbf{P}^1 (*i.e.* maps from \mathbf{P}^1 into a Grassmannian) have a natural stratification, corresponding to the fact that vector bundles over \mathbf{P}^1 decompose into a sum of line bundles, and there are different ways to distribute the c_1 among the various line bundles.

The strata we have seen in the last few examples, are precisely strata of corresponding Quot schemes, and the condition on the exceptional set we described in the first example, is precisely the right condition to reproduce Quot schemes. In other words, the last few examples have implicitly described how Quot schemes arise physically in nonabelian gauged linear sigma models.

The fact that Quot schemes parametrize maps from \mathbf{P}^1 into Grassmannians, and hyperquot schemes parametrize maps from \mathbf{P}^1 into flag manifolds, is well-known to mathemati-

cians, see for example⁹ [38, 40, 41, 42, 44, 47, 48]. What is new here is the understanding that gauged linear sigma models for Grassmannians also see precisely Quot's, and not some other, distinct (possibly birational) spaces instead.

See [51] for more information on Quot schemes.

We have spent a great deal of time studying maps into Grassmannians, but there are analogous stories in other nonabelian GLSM's. For example, maps into partial flag manifolds are described mathematically by hyperquot schemes, which parametrize flags of bundles. The physical analysis of the GLSM moduli spaces in GLSM's for flag manifolds is very similar to what we have done here in GLSM's for Grassmannians, and the conclusion is analogous: GLSM's for flag manifolds see moduli spaces of maps given by hyperquot schemes, as one would naively expect. We will outline the analysis and describe an example next, to support this claim.

5.3.6 Flag manifolds

Let us briefly outline the analysis of linear sigma model moduli spaces for flag manifolds.

Recall the linear sigma model describes $F(k_1, k_2, \dots, k_n, N)$ as a GIT quotient

$$\left(\mathbf{C}^{k_1 k_2} \times \mathbf{C}^{k_2 k_3} \times \dots \times \mathbf{C}^{k_n N} \right) // (GL(k_1) \times GL(k_2) \times \dots GL(k_n))$$

Thus, instead of a single vector bundle ϕ^S , we now have a collection of vector bundles $\phi^* S_i$, of rank k_i , one for each factor in the gauge group of the linear sigma model, together with inclusion maps

$$\phi^* S_1 \hookrightarrow \phi^* S_2 \hookrightarrow \dots \hookrightarrow \phi^* S_n \hookrightarrow \mathcal{O}_{\mathbf{P}^1}^N$$

Because our worldsheet is \mathbf{P}^1 , each of these vector bundles is specified topologically by their first Chern class, denoted $d_i = c_1(\phi^* S_i^\vee)$, and each vector bundle will completely split over \mathbf{P}^1 , yielding a stratification of the moduli space, which we claim can be understood globally as a hyperquot scheme.

To fix conventions, we say that a map from \mathbf{P}^1 into the flag manifold $F(k_1, \dots, k_n, N)$ has degree (d_1, \dots, d_n) if $c_1(\phi^* S_i^\vee) = d_i$.

In these conventions, the zero modes of the bifundamentals in the $(\mathbf{k}_{i-1}, \overline{\mathbf{k}}_i)$ representation are holomorphic sections

$$H^0(\mathbf{P}^1, \phi^* S_{i-1}^\vee \otimes \phi^* S_i)$$

(in conventions where $\phi^* S_{i+1} = \mathcal{O}^N$). Also, note that since the $\phi^* S_i$'s are vector bundles, we do not need to require

$$c_1(\phi^* S_{i-1}^\vee \otimes \phi^* S_i) \geq 0$$

⁹This is merely a sample list of references. It is not intended to be either complete or comprehensive.

for the moduli space to be nonempty.

5.3.7 Flag manifold example

Let us consider the example of maps of degree $(2, 1)$ from \mathbf{P}^1 into $F(1, 2, N)$. Such a map is specified by two vector bundles over \mathbf{P}^1 $\phi^*S_1^\vee$, $\phi^*S_2^\vee$ of c_1 equal to 2, 1, and ranks 1, 2, respectively.

This information uniquely specifies $\phi^*S_1^\vee = \mathcal{O}(2)$. Up to irrelevant ordering, this also uniquely specifies that $\phi^*S_2^\vee = \mathcal{O}(1) \oplus \mathcal{O}$.

The zero modes of the bifundamentals are given as follows. The bifundamentals in the $(1, \overline{2})$ representation of $U(1) \times U(2)$ have zero modes counted by

$$H^0(\mathbf{P}^1, \mathcal{O}(2-1) \oplus \mathcal{O}(2-0)) = H^0(\mathbf{P}^1, \mathcal{O}(1) \oplus \mathcal{O}(2)) = \mathbf{C}^2 \times \mathbf{C}^3 = \mathbf{C}^5$$

The remaining bifundamentals, in the $(2, \overline{\mathbf{N}})$ representation, have zero modes counted by

$$H^0(\mathbf{P}^1, (\mathcal{O}(1) \oplus \mathcal{O}) \otimes \mathcal{O}^N) = \mathbf{C}^{2N} \times \mathbf{C}^N = \mathbf{C}^{3N}$$

The linear sigma model moduli space will then be $(\mathbf{C}^5 \times \mathbf{C}^{3N})//G$ for some G . In the present case, G will be a product of two factors, determined by the $U(1)$ and $U(2)$ factors in the gauge group of the original linear sigma model. One of the factors in G will be all of $GL(1)$, the automorphisms of $\mathcal{O}(2)$, which acts on the \mathbf{C}^5 in $\mathbf{C}^5 \times \mathbf{C}^{3N}$ with weight one. The other factor, determined by the $U(2)$ of the linear sigma model gauge group, will be a non-reductive group defined as automorphisms of $\mathcal{O}(1) \oplus \mathcal{O}(0)$, an extension of $GL(1)^2$, as in section 5.3.3. This group, call it G' , acts on both the \mathbf{C}^5 and the \mathbf{C}^{3N} factors. So, the linear sigma model moduli space has the form

$$(\mathbf{C}^5 \times \mathbf{C}^{3N}) // (GL(1) \times G')$$

Moreover, this space is exactly the pertinent hyperquot scheme, giving us empirical confirmation of the claim that linear sigma model moduli spaces for maps from \mathbf{P}^1 into Grassmannians and flag manifolds should be precisely Quot schemes and hyperquot schemes, respectively.

6 Conclusions

In this paper, we have described gauged linear sigma models for flag manifolds and their properties. In particular, we have accomplished three things. First, we have described the

gauged linear sigma models that describe partial flag manifolds, and outlined aspects relevant to $(0,2)$ GLSM's, which does not seem to have been in the literature previously.

Second, we have described properties of gauged linear sigma models for complete intersections in flag manifolds. After reviewing the Calabi-Yau conditions for a flag manifold, we have discussed various examples of phenomena in which different geometric Kähler phases of a given GLSM are not related by birational transformation. We have discussed both Calabi-Yau and non-Calabi-Yau examples of this phenomenon, and also proposed a mathematical understanding that should be the right notion to replace ‘birational’ in this context, namely, Kuznetsov’s homological projective duality. Although nearly every example discussed in this paper was a nonabelian GLSM, we also outline how similar phenomena happen in abelian GLSM’s, which will be studied in greater detail (together with Kuznetsov’s homological projective duality) in [9].

Finally, we have outlined how quantum cohomology of flag manifolds arises within gauged linear sigma models. In particular, we describe linear sigma model moduli spaces of maps into Grassmannians and flag manifolds. The mathematically natural notion is that those spaces are Quot and hyperquot schemes, and we describe how to see them explicitly in the physics of nonabelian GLSM’s.

7 Acknowledgements

We would like to thank A. Bertram, T. Braden, A. Caldararu, L. Chen, I. Ciocan-Fontanine, J. Distler, M. Gross, S. Hellerman, S. Katz, and especially A. Knutson, A. Kuznetsov, and T. Pantev for useful conversations. This paper began as a project with A. Knutson, and later grew in part out of the related work [9], done in collaboration with A. Caldararu, J. Distler, S. Hellerman, and T. Pantev. The work of R. Donagi was partially supported by NSF grants DMS-0612992 and Focused Research Grant DMS-0139799 for “The geometry of superstrings.”

References

- [1] E. Witten, “Phases of $N=2$ theories in two dimensions,” Nucl Phys. **B403** (1993) 159-222, [hep-th/9301042](#).
- [2] T. Pantev, E. Sharpe, “GLSM’s for gerbes (and other toric stacks),” Adv. Theor. Math. Phys. **10** (2006) 77-121, [hep-th/0502053](#).
- [3] E. Witten, “The Verlinde algebra and the cohomology of the Grassmannian,” [hep-th/9312104](#).

- [4] K. Hori, D. Tong, “Aspects of non-Abelian gauge dynamics in two-dimensional $N = (2, 2)$ theories,” [hep-th/0609032](#).
- [5] S. Hellerman, A. Henriques, T. Pantev, E. Sharpe, M. Ando, “Cluster decomposition, T-duality, and gerby CFT’s,” [hep-th/0606034](#).
- [6] A. Kuznetsov, “Homological projective duality,” [math.AG/0507292](#).
- [7] A. Kuznetsov, “Derived categories of quadric fibrations and intersections of quadrics,” [math.AG/0510670](#).
- [8] A. Kuznetsov, “Homological projective duality for Grassmannians of lines,” [math.AG/0610957](#).
- [9] A. Caldararu, J. Distler, S. Hellerman, T. Pantev, and E. Sharpe, “Non-birational twisted derived equivalences in abelian GLSM’s,” to appear.
- [10] Y. Manin, *Gauge field theory and complex geometry*, Springer-Verlag, 1988.
- [11] T. Hubsch, *Calabi-Yau manifolds: a bestiary for physicists*, World Scientific, 1992.
- [12] T. Pantev, E. Sharpe, “Notes on gauging noneffective group actions,” [hep-th/0502027](#).
- [13] T. Pantev, E. Sharpe, “String compactifications on Calabi-Yau stacks,” Nucl. Phys. **B733** (2006) 233-296, [hep-th/0502044](#).
- [14] E. Sharpe, “Derived categories and stacks in physics,” [hep-th/0608056](#).
- [15] N. Seiberg, “Electric-magnetic duality in supersymmetric non-Abelian gauge theories,” [hep-th/9411149](#).
- [16] W. V. D. Hodge, D. Pedoe, *Methods of Algebraic Geometry*, volume 1, Cambridge University Press, 1947.
- [17] R. Bott, L. Tu, *Differential forms in algebraic topology*, Springer-Verlag, 1982.
- [18] I. Klebanov, M. Strassler, “Supergravity and a confining gauge theory: duality cascades and chiral-symmetry-breaking resolution of naked singularities,” [hep-th/0007191](#).
- [19] M. Strassler, “The duality cascade,” [hep-th/0505153](#).
- [20] A. Corti, M. Reid, “Weighted Grassmannians,” [math.AG/0206011](#).
- [21] J. Distler, “Notes on (0,2) superconformal field theories,” [hep-th/9502012](#).
- [22] V. Batyrev, I. Ciocan-Fontanine, B. Kim, D. van Straten, “Mirror symmetry and toric degenerations of partial flag manifolds,” [math.AG/9803108](#).

- [23] V. Batyrev, I. Ciocan-Fontanine, B. Kim, D. van Straten, “Conifold transitions and mirror symmetry for Calabi-Yau complete intersections in Grassmannians,” Nucl. Phys. **B514** (1998) 640-666, [alg-geom/9710022](#).
- [24] L. Borisov, A. Caldararu, “The Pfaffian-Grassmannian derived equivalence,” [math.AG/0608404](#).
- [25] J. Harris, L. Tu, “On symmetric and skew-symmetric determinantal varieties,” Topology **23** (1984) 71-84.
- [26] A. Caldararu, “Derived categories of twisted sheaves on elliptic threefolds,” [math.AG/0012083](#).
- [27] A. Caldararu, private communication.
- [28] T. Pantev, private communication.
- [29] D. Orlov, “Derived categories of coherent sheaves and triangulated categories of singularities,” [math.AG/0503632](#).
- [30] E. Sharpe, “D-branes, derived categories, and Grothendieck groups,” Nucl. Phys. **B561** (1999) 433-450, [hep-th/9902116](#).
- [31] M. Douglas, “D-branes, categories, and $\mathcal{N} = 1$ supersymmetry,” J. Math. Phys. **42** (2001) 004, [hep-th/0104147](#).
- [32] A. Beauville, R. Donagi, “La variété des droits d’une hypersurface cubique de dimension 4,” C.R. Acad. Sci. Paris. Ser. I Math. 1985.
- [33] A. Kuznetsov, private communication, March 10, 2007.
- [34] A. Kuznetsov, private communication, April 3, 2007.
- [35] M. Gross, private communication, September 27, 2006.
- [36] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, John Wiley & Sons, New York, 1978.
- [37] S. Mukai, “Moduli of vector bundles on K3 surfaces, and symplectic manifolds,” Sugaku Expositions **1** (1988) 139-174.
- [38] A. Bertram, G. Daskalopoulos, and R. Wentworth, “Gromov invariants for holomorphic maps from Riemann surfaces to Grassmannians,” J. Amer. Math. Soc. **9** (1996) 529-571.
- [39] A. Ashtashkevich, V. Sadov, “Quantum cohomology of partial flag manifolds $F_{n_1 \dots n_k}$,” Comm. Math. Phys. **170** (1995) 503-528.

- [40] I. Ciocan-Fontanine, “On quantum cohomology rings of partial flag varieties,” *Duke Math. J.* **98** (1999) 485-524.
- [41] I. Ciocan-Fontanine, “The quantum cohomology ring of flag varieties,” *Trans. Amer. Math. Soc.* **351** (1999) 2695-2729.
- [42] I. Ciocan-Fontanine, “Quantum cohomology of flag varieties,” *Int. Math. Res. Notices* 1995, no. 6, 263-277.
- [43] B. Kim, “Quantum cohomology of partial flag manifolds and a residue formula for their intersection pairings,” *Int. Math. Res. Notices* 1995, no. 1, 1-16.
- [44] B. Kim, “Quot schemes for flags and Gromov invariants for flag varieties,” [alg-geom/9512003](#).
- [45] A. Bertram, I. Ciocan-Fontanine, B. Kim, “Two proofs of a conjecture of Hori and Vafa,” *Duke Math. J.* **126** (2005) 101-136, [math.AG/0304403](#).
- [46] A. Bertram, I. Ciocan-Fontanine, B. Kim, “Gromov-Witten invariants for abelian and nonabelian quotients,” [math.AG/0407254](#).
- [47] L. Chen, “Poincare polynomials of hyperquot schemes,” [math.AG/0003077](#).
- [48] S. A. Stromme, “On parametrized rational curves in Grassmann varieties,” pp. 251-272 in *Space Curves*, *Lect. notes in math.* 1266, ed. F. Ghione, C. Peskine, E. Sernesi, Springer-Verlag, 1987.
- [49] B. Lian, C.-H. Liu, K. Liu, S.-T. Yau, “The S^1 fixed points in Quot schemes and mirror principle computations,” [math/0111256](#).
- [50] D. Morrison, R. Plesser, “Summing the instantons: quantum cohomology and mirror symmetry in toric varieties,” *Nucl. Phys.* **B440** (1995) 279-354, [hep-th/9412236](#).
- [51] D. Huybrechts, M. Lehn, *The geometry of moduli spaces of sheaves*, Vieweg, 1997.